ECLECTIC EDUCATIONAL SERIES.

TREATISE

ON

GEOMETRY

AND

TRIGONOMETRY:

FOR

COLLEGES, SCHOOLS AND PRIVATE STUDENTS.

WRITTEN FOR THE MATHEMATICAL COURSE OF

JOSEPH RAY, M. D.,

BY

ELI T. TAPPAN, M. A.,

PROFESSOR OF MATHEMATICS, OHIO UNIVERSITY.

CINCINNATI:

WILSON, HINKLE & CO.

PHIL'A: CLAXTON, REMSEN & HAFFELFINGER.

NEW YORK: CLARK & MAYNARD.

THE BEST AND CHEAPEST.

RAY'S

MATHEMATICAL SERIES.

Ray's Primary Arithmetic: Simple Mental Lessons and Tables. For little Learners.

Ray's Intellectual Arithmetic: the most interesting and valuable Arithmetic extant.

Ray's Rudiments of Arithmetic: combining mental and practical exercises. For beginners.

Ray's Practical Arithmetic: a full and practical treatise on the inductive and analytic methods of instruction.

Ray's Higher Arithmetic: the principles of Arithmetic analyzed and practically applied.

Ray's Test Examples: three thousand practical problems for the slate or blackboard. For drill exercises and review.

Ray's New Elementary Algebra: a simple, thorough, and progressive elementary treatise. For Schools and Academies.

Ray's New Higher Algebra: a progressive, lucid, and comprehensive work. For advanced Students and for Colleges.

Ray's Elements of Geometry: a comprehensive work on Plane and Solid Geometry, with numerous practical exercises.

Ray's Geometry and Trigonometry: Plane and Spherical Trigonometry, with their applications; also a complete set of Logarithmic tables, carefully corrected.

Ray's Differential and Integral Calculus: in course of preparation, and to be published during the present year.

To be followed, at an early day, by other works, forming a complete Mathematical Course for Schools and Colleges.

ELECTROTYPED AT THE FRANKLIN TYPE FOUNDRY, CINCINNATI.

PREFACE.

THE science of Elementary Geometry, after remaining nearly stationary for two thousand years, has, for a century past, been making decided progress. This is owing, mainly, to two causes: discoveries in the higher mathematics have thrown new light upon the elements of the science; and the demands of schools, in all enlightened nations, have called out many works by able mathematicians and skillful teachers.

Professor Hayward, of Harvard University, as early as 1825, defined parallel lines as lines having the same direction. Euclid's definitions of a straight line, of an angle, and of a plane, were based on the idea of direction, which is, indeed, the essence of form. This thought, employed in all these leading definitions, adds clearness to the science and simplicity to the study. In the present work, it is sought to combine these ideas with the best methods and latest discoveries in the science.

By careful arrangement of topics, the theory of each class of figures is given in uninterrupted connection. No attempt is made to exclude any method of demonstration, but rather to present examples of all.

The books most freely used are, "Cours de géométrie élémentaire, par A. J. H. Vincent et M. Bourdon;" "Géométrie théorique et pratique, etc., par H. Sonnet;" "Die reine elementar-mathematik, von Dr. Martin Ohm;" and "Treatise on Geometry and its application to the Arts, by Rev. D. Lardner."

The subject is divided into chapters, and the articles are numbered continuously through the entire work. The convenience of this arrangement for purposes of reference, has caused it to be adopted by a large majority of writers upon Geometry, as it had been by writers on other scientific subjects.

In the chapters on Trigonometry, this science is treated as a branch of Algebra applied to Geometry, and the trigonometrical functions are defined as ratios. This method has the advantages of being more simple and more brief, yet more comprehensive than the ancient geometrical method.

For many things in these chapters, credit is due to the works of Mr. I. Todhunter, M. A., St. John's College, Cambridge.

The tables of logarithms of numbers and of sines and tangents have been carefully read with the corrected edition of Callet, with the tables of Dr. Schrön, and with those of Babbage.

ELI T. TAPPAN.

OHIO UNIVERSITY, *Jan.* 1, 1868.

CONTENTS.

PART FIRST.—INTRODUCTORY.

CHAPTER I.

PRELIMINARY.

CHAPTER II.

THE SUBJECT STATED.

PART SECOND.—PLANE GEOMETRY.

CHAPTER III.

STRAIGHT LINES.

CHAPTER IV.

CIRCUMFERENCES.

CHAPTER V.

TRIANGLES.

CHAPTER VI.

QUADRILATERALS.

CHAPTER VII.

POLYGONS.

CHAPTER VIII.

CIRCLES.

PART THIRD.—GEOMETRY OF SPACE.

CHAPTER IX.

STRAIGHT LINES AND PLANES.

CHAPTER X.

POLYEDRONS.

CHAPTER XI.

SOLIDS OF REVOLUTION.

PART FOURTH.—TRIGONOMETRY.

CHAPTER XII.

PLANE TRIGONOMETRY.

CHAPTER XIII.

SPHERICAL TRIGONOMETRY.

CHAPTER XIV.

LOGARITHMS.

TABLES.

ELEMENTS

OF

GEOMETRY.

CHAPTER I.—PRELIMINARY.

Article 1. BEFORE the student begins the study of geometry, he should know certain principles and definitions, which are of frequent use, though they are not peculiar to this science. They are very briefly presented in this chapter.

LOGICAL TERMS.

2. Every statement of a principle is called a PROPOSITION.

Every proposition contains the subject of which the assertion is made, and the property or circumstance asserted.

When the subject has some condition attached to it, the proposition is said to be conditional.

The subject, with its condition, if it have any, is the HYPOTHESIS of the proposition, and the thing asserted is the CONCLUSION.

Each of two propositions is the CONVERSE of the other, when the two are such that the hypothesis of either is the conclusion of the other.

3. A proposition is either *theoretical*, that is, it declares that a certain property belongs to a certain thing; or it is *practical*, that is, it declares that something can be done.

Propositions are either *demonstrable*, that is, they may be established by the aid of reason; or they are *indemonstrable*, that is, so simple and evident that they can not be made more so by any course of reasoning.

A THEOREM is a demonstrable, theoretical proposition.

A PROBLEM is a demonstrable, practical proposition.

An AXIOM is an indemonstrable, theoretical proposition.

A POSTULATE is an indemonstrable, practical proposition.

A proposition which flows, without additional reasoning, from previous principles, is called a COROLLARY. This term is also frequently applied to propositions, the demonstration of which is very brief and simple.

4. The reasoning by which a proposition is proved is called the DEMONSTRATION.

The explanation how a thing is done constitutes the SOLUTION of a problem.

A DIRECT DEMONSTRATION proceeds from the premises by a regular deduction.

An INDIRECT DEMONSTRATION attains its object by showing that any other hypothesis or supposition than the one advanced would involve a contradiction, or lead to an impossible conclusion. Such a conclusion may be called absurd, and hence the Latin name of this method of reasoning—*reductio ad absurdum.*

A work on Geometry consists of definitions, propositions, demonstrations, and solutions, with introductory or explanatory remarks. Such remarks sometimes have the name of scholia.

5. Remark.—The student should learn each proposition, so as to state separately the hypothesis and the conclusion, also the condition, if any. He should also learn, at each demonstration, whether it is direct or indirect; and if indirect, then what is the false hypothesis and what is the absurd conclusion. It is a good exercise to state the converse of a proposition.

In this work the propositions are first enounced in general terms. This general enunciation is usually followed by a particular statement of the principle, as a fact, referring to a diagram. Then follows the demonstration or solution. In the latter part of the work these steps are frequently shortened.

The student is advised to conclude every demonstration with the general proposition which he has proved.

The student meeting a reference, should be certain that he can state and apply the principle referred to.

GENERAL AXIOMS.

6. *Quantities which are each equal to the same quantity, are equal to each other.*

7. *If the same operation be performed upon equal quantities, the results will be equal.*

For example, if the same quantity be separately added to two equal quantities, the sums will be equal.

8. *If the same operation be performed upon unequal quantities, the results will be unequal.*

Thus, if the same quantity be subtracted from two unequal quantities, the remainder of the greater will exceed the remainder of the less.

9. *The whole is equal to the sum of all the parts.*

10. *The whole is greater than a part.*

EXERCISE.

11. What is the hypothesis of the first axiom?

What is the subject of the first axiom?

What is the condition of the first axiom?

What is the conclusion of the first axiom?

Give an example of this axiom.

RATIO AND PROPORTION

12. All mathematical investigations are conducted by comparing quantities, for we can form no conception of any quantity except by comparison.

13. In the comparison of one quantity with another, the relation may be noted in two ways: either, first, how much one exceeds the other; or, second, how many times one contains the other.

The result of the first method is the difference between the two quantities; the result of the second is the RATIO of one to the other.

Every ratio, as it expresses "how many times" one quantity contains another, is a number. That a ratio and a number are quantities of the same kind, is further shown by comparing them; for we can find their sum, their difference, or the ratio of one to the other.

When the division can be exactly performed, the ratio is a whole number; but it may be a fraction, or a radical, or some other number incommensurable with unity.

14. The symbols of the quantities from whose comparison a ratio is derived, are frequently retained in its expression. Thus,

The ratio of a quantity represented by a to another represented by b, may be written $\frac{a}{b}$.

A ratio is usually written $a : b$, and is read, a is to b.

This retaining of the symbols is merely for convenience, and to show the derivation of the ratio; for a ratio may be expressed by a single figure, or by any other symbol, as 2, m, $\sqrt{3}$, or π. But since every ratio is a number, therefore, when a ratio is thus expressed by means of two terms, they must be understood to represent two numbers having the same relation as the given quantities.

The second term is the standard or unit with which the first is compared.

So, when the ratio is expressed in the form of a fraction, the first term, or ANTECEDENT, becomes the numerator, and the second, or CONSEQUENT, the denominator.

15. A PROPORTION is the equality of two ratios, and is generally written,

$$a : b :: c : d,$$

and is read, a is to b as c is to d,

but it is sometimes written,

$$a : b = c : d,$$

or it may be, $\frac{a}{b} = \frac{c}{d}$,

all of which express the same thing: that a contains b exactly as often as c contains d.

The first and last terms are the EXTREMES, and the second and third are the MEANS of a proportion.

The fourth term is called the FOURTH PROPORTIONAL of the other three.

A series of equal ratios is written,

$$a : b :: c : d :: e : f, \text{ etc.}$$

When a series of quantities is such that the ratio of each to the next following is the same, they are written,

$$a : b : c : d, \text{ etc.}$$

Here, each term, except the first and last, is both antecedent and consequent. When such a series consists of three terms, the second is the MEAN PROPORTIONAL of the other two.

16. Proposition.—*The product of the extremes of any proportion is equal to the product of the means.*

For any proportion, as

$$a : b :: c : d,$$

is the equation of two fractions, and may be written,

$$\frac{a}{b} = \frac{c}{d}.$$

Multiplying these equals by the product of the denominators, we have (7)

$$a \times d = b \times c,$$

or the product of the extremes equal to the product of the means.

17. Corollary.—The square of a mean proportional is equal to the product of the extremes. A mean proportional of two quantities is the square root of their product.

18. Proposition.—*When the product of two quantities is equal to the product of two others, either two may be the extremes and the other two the means of a proportion.*

Let $a \times d = b \times c$ represent the equal products.

If we divide by b and d, we have

$$\frac{a}{b} = \frac{c}{d}; \text{ or, } a : b :: c : d. \qquad \text{(1st.)}$$

If we divide by c and d, we have

$$\frac{a}{c} = \frac{b}{d}; \text{ or, } a : c :: b : d. \qquad \text{(2d.)}$$

If we arrange the equal products thus:

$$b \times c = a \times d,$$

and then divide by a and c, we have

$$b : a :: d : c. \qquad (3d.)$$

By similar divisions, the student may produce five other arrangements of the same quantities in proportion.

19. Proposition.—*The order of the terms may be changed without destroying the proportion, so long as the extremes remain extremes, or both become means.*

Let $a : b :: c : d$ represent the given proportion.

Then (16), we have $a \times d = b \times c$. Therefore (18), a and d may be taken as either the extremes or the means of a new proportion.

20. When we say the first term is to the third as the second is to the fourth, the proportion is taken by *alternation*, as in the second case, Article 18.

When we say the second term is to the first as the fourth is to the third, the proportion is taken *inversely*, as in the third case.

21. Proposition.—*Ratios which are equal to the same ratio are equal to each other.*

This is a case of the first axiom (6).

22. Proposition.—*If two quantities have the same multiplier, the multiples will have the same ratio as the given quantities.*

Let a and b represent any two quantities, and m any multiplier. Then the identical equation,

$$m \times a \times b = m \times b \times a,$$

gives the proportion,

$$m \times a : m \times b :: a : b \text{ (18).}$$

23. Proposition.—*In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Let $a : b :: c : d :: e : f :: g : h$, etc., represent the equal ratios.

Therefore (16),
$$a \times d = b \times c$$
$$a \times f = b \times e$$
$$a \times h = b \times g$$
To these add
$$a \times b = b \times a$$
$$a \times (b+d+f+h) = b \times (a+c+e+g).$$

Therefore (18),
$$a+c+e+g : b+d+f+h :: a : b.$$

This is called proportion by COMPOSITION.

24. Proposition.— *The difference between the first and second terms of a proportion is to the second, as the difference between the third and fourth is to the fourth.*

The given proportion,
$$a : b :: c : d,$$
may be written,
$$\frac{a}{b} = \frac{c}{d}.$$

Subtract the identical equation,
$$\frac{b}{b} = \frac{d}{d}.$$

The remaining equation,
$$\frac{a-b}{b} = \frac{c-d}{d},$$
may be written, $a-b : b :: c-d : d.$

This is called proportion by DIVISION.

25. Proposition.—*If four quantities are in proportion, their same powers are in proportion, also their same roots.*

Thus, if we have $a : b :: c : d$,

then, $a^2 : b^2 :: c^2 : d^2$;

also, $\sqrt{a} : \sqrt{b} :: \sqrt{c} : \sqrt{d}.$

These principles are corollaries of the second general axiom (7), since a proportion is an equation.

CHAPTER II.

THE SUBJECT STATED.

26. WE know that every material object occupies a portion of space, and has extent and form.

For example, this book occupies a certain space; it has a definite extent, and an exact form. These properties may be considered separate, or abstract from all others. If the book be removed, the space which it had occupied remains, and has these properties, extent and form, and none other.

27. Such a limited portion of space is called a solid.

Be careful to distinguish the geometrical solid, which is a portion of space, from the solid body which occupies space.

Solids may be of all the varieties of extent and form that are found in nature or art, or that can be imagined.

28. The limit or boundary which separates a solid from the surrounding space is a surface. A surface is like a solid in having only these two properties, extent and form; but a surface differs from a solid in having no thickness or depth, so that a solid has one kind of extent which a surface has not.

As solids and surfaces have an abstract existence, without material bodies, so two solids may occupy the same space, entirely or partially. For example, the position which has been occupied by a book, may be now occupied by a block of wood. The solids represented by the book and block may occupy at once, to some extent, the same space. Their surfaces may meet or cut each other.

29. The limits or boundaries of a surface are lines. The intersection of two surfaces, being the limit of the parts into which each divides the other, is a line.

A line has these two properties only, extent and form; but a surface has one kind of extent which a line has not: a line differs from a surface in the same way that a surface does from a solid. A line has neither thickness nor breadth.

30. The ends or limits of a line are points. The intersections of lines are also points. A point is unlike either lines, surfaces, or solids, in this, that it has neither extent nor form.

31. As one line may be met by any number of others, and a surface cut by any number of others; so a line may have any number of points, and a surface any number of lines and points. And a solid may have any number of intersecting surfaces, with their lines and points.

DEFINITIONS.

32. These considerations have led to the following definitions:

A POINT has only position, without extent.

A LINE has length, without breadth or thickness.

A SURFACE has length and breadth, without thickness.

A SOLID has length, breadth, and thickness.

33. A line may be measured only in one way, or, it may be said a line has only one dimension. A surface has two, and a solid has three dimensions. We can not conceive of any thing of more than three dimensions. Therefore, every thing which has extent and form belongs to one of these three classes.

The extent of a line is called its LENGTH; of a surface, its AREA; and of a solid, its VOLUME.

34. Whatever has only extent and form is called a MAGNITUDE.

GEOMETRY is the science of magnitude.

Geometry is used whenever the size, shape, or position of any thing is investigated. It establishes the principles upon which all measurements are made. It is the basis of Surveying, Navigation, and Astronomy.

In addition to these uses of Geometry, the study is cultivated for the purpose of training the student's powers of language, in the use of precise terms; his reason, in the various analyses and demonstrations; and his inventive faculty, in the making of new solutions and demonstrations.

THE POSTULATES.

35. Magnitudes may have any extent. We may conceive lines, surfaces, or solids, which do not extend beyond the limits of the smallest spot which represents a point; or, we may conceive them of such extent as to reach across the universe. The astronomer knows that his lines reach to the stars, and his planes extend beyond the sun. These ideas are expressed in the following

Postulate of Extent.—*A magnitude may be made to have any extent whatever.*

36. Magnitudes may, in our minds, have any form, from the most simple, such as a straight line, to that of the most complicated piece of machinery. We may conceive of surfaces without solids, and of lines without surfaces.

It is a useful exercise to imagine lines of various forms, extending not only along the paper or blackboard, but across the room. In the same way, surfaces and solids may be conceived of all possible forms.

The form of a magnitude consists in the relative position of the parts, that is, in the relative directions of the points. Every change of form consists in changing the relative directions of the points of the figure.

Every geometrical conception, however simple or complex, is composed of only two kinds of elementary thoughts—directions and distances. The directions determine its form, and the distances its extent.

Postulate of Form.—*The points of a magnitude may be made to have from each other any directions whatever, thus giving the magnitude any conceivable form.*

These two are all the postulates of geometry. They rest in the very ideas of space, form, and magnitude.

37. Magnitudes which have the same form while they differ in extent, are called SIMILAR.

Any point, line, or surface in a figure, and the similarly situated point, line, or surface in a similar figure, are called HOMOLOGOUS.

Magnitudes which have the same extent, while they differ in form, are called EQUIVALENT.

MOTION AND SUPERPOSITION.

38. The postulates are of constant use in geometrical reasoning.

Since the parts of a magnitude may have any position, they may change position. By this idea of motion, the mutual derivation of points, lines, surfaces, and solids may be explained.

The path of a point is a line, the path of a line may be a surface, and the path of a surface may be a solid. The time or rate of motion is not a subject of geometry, but the path of any thing is itself a magnitude.

39. By the idea of motion, one magnitude may be mentally applied to another, and their form and extent compared.

This is called the method of superposition, and is the most simple and useful of all the methods of demonstration used in geometry. The student will meet with many examples.

EQUALITY.

40. When two *equal* magnitudes are compared, it is found that they may coincide; that is, each contains the other. Since they coincide, every part of one will have its corresponding equal and coinciding part in the other, and the parts are arranged the same in both.

Conversely, if two magnitudes are composed of parts respectively equal and similarly arranged, one may be applied to the other, part by part, till the wholes coincide, showing the two magnitudes to be equal.

Each of the above convertible propositions has been stated as an axiom, but they appear rather to constitute the definition of equality.

FIGURES.

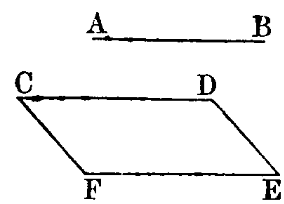
41. Any magnitude or combination of magnitudes which can be accurately described, is called a geometrical FIGURE.

Figures are represented by diagrams or drawings, and such representations are, in common language, called figures. A small spot is commonly called a point, and a long mark a line. But these have not only extent and form, but also color, weight, and other properties; and, therefore, they are not *geometrical* points and lines.

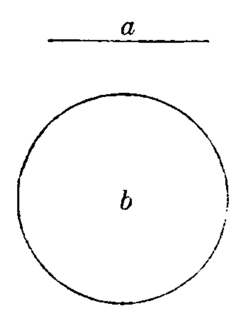
It is the more important to remember this distinction, since the point and line made with chalk or ink are constantly used to represent to the eye true mathematical points and lines.

42. The figure which is the subject of a proposition, together with all its parts, is said to be GIVEN. The additions to the figure made for the purpose of demonstration or solution, constitute the CONSTRUCTION.

43. In the diagrams in this work, points are designated by capital letters. Thus, the points A and B are at the extremities of the line.



Figures are usually designated by naming some of their points, as the line AB, and the figure CDEF, or simply the figure DF.



When it is more convenient to designate a figure by a single letter, the small letters are used. Thus, the line *a*, or the figure *b*.

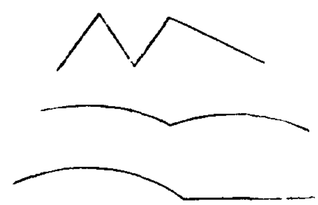
LINES.

44. A STRAIGHT LINE is one which has the same direction throughout its whole extent.

A straight line may be regarded as the path of a point moving in one direction, turning neither up nor down, to the right or left.

45. A CURVED LINE is one which constantly changes its direction. The word *curve* is used for a *curved line.*

46. A line composed of straight lines, is called BROKEN. A line may be composed of curves, or of both curved and straight parts.



THE STRAIGHT LINE.

47. Problem.—*A straight line may be made to pass through any two points.*

48. Problem.—*There may be a straight line from any point, in any direction, and of any extent.*

These two propositions are corollaries of the postulates.

49. From a point, straight lines may extend in all directions. But we can not conceive that two separate straight lines can have the same direction from a common point. This impossibility is expressed by the following

Axiom of Direction.—*In one direction from a point, there can be only one straight line.*

50. Corollary.—From one point to another, there can be only one straight line

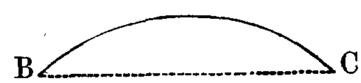
51. Theorem.—*If a straight line have two of its points common with another straight line, the two lines must coincide throughout their mutual extent.*

For, if they could separate, there would be from the point of separation two straight lines having the same direction, which is impossible (49).

52. Corollary.—Two fixed points, or one point and a certain direction, determine the position of a straight line.

53. If a straight line were turned upon two of its points as fixed pivots, no part of the line would change place. So any figure may revolve about a straight line, while the position of the line remains unchanged.

This property is peculiar to the straight line. If the curve BC were to revolve upon the two points B and C as pivots, then the straight line connecting these points would remain at rest, and the curve would revolve about it.



A straight line about which any thing revolves, is called its AXIS.

54. Axiom of Distance.—*The straight line is the shortest which can join two points.*

Therefore, the distance from one point to another is reckoned along a straight line.

55. There have now been given two postulates and two axioms. The science of geometry rests upon these four simple truths.

The possibility of every figure defined, and the truth of every problem, depend upon the postulates.

Upon the postulates, with the axioms, is built the demonstration of every principle.

SURFACES.

56. Surfaces, like lines, are classified according to their uniformity or change of direction.

A PLANE is a surface which never varies in direction.

A CURVED SURFACE is one in which there is a change of direction at every point.

THE PLANE.

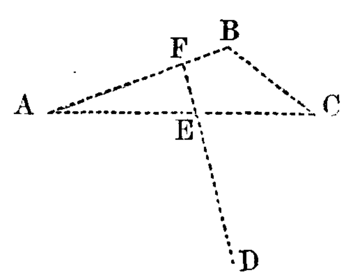
57. The plane surface and the straight line have the same essential character, sameness of direction. The plane is straight in every direction that it has.

A straight line and a plane, unless the extent be specified, are always understood to be of indefinite extent.

58. Theorem.—*A straight line which has two points in a plane, lies wholly in it, so far as they both extend.*

For if the line and surface could separate, one or the other would change direction, which by their definitions is impossible.

59. Theorem.—*Two planes having three points common, and not in the same straight line, coincide so far as they both extend.*



Let A, B, and C be three points which are not in one straight line, and let these points be common to two planes, which may be designated by the letters *m* and *p*. Let a straight line pass through the points A and B, a second through B and C, and a third through A and C.

Each of these lines (58) lies wholly in each of the planes *m* and *p*. Now it is to be proved that any point D, in the plane *m*, must also be in the plane *p*.

Let a line extend from D to some point of the line AC, as E. The points D and E being in the plane *m*, the whole line DE must be in that plane; and, therefore, if produced across the inclosed surface ABC, it will meet one of the other lines AB, BC, which also lie in that plane, say, at the point F. But the points F and E are both in the plane *p*. Therefore, the whole line FD, including the point D, is in the plane *p*.

In the same manner, it may be shown that any point which is in one plane, is also in the other, and therefore the two planes coincide.

60. Corollary.—Three points not in a straight line, or a straight line and a point out of it, fix the position of a plane.

61. Corollary.—That part of a plane on one side of any straight line in it, may revolve about the line till it meets the other part, when the two will coincide (53).

EXERCISES.

62. When a mechanic wishes to know whether a line is straight, he may apply another line to it, and observe if they coincide.

In order to try if a surface is plane, he applies a straight rule to it in many directions, observing whether the two touch throughout.

The mason, in order to obtain a plain surface to his marble, applies another surface to it, and the two are ground together until all unevenness is smoothed away, and the two touch throughout.

What geometrical principle is used in each of these operations?

In a diagram two letters suffice to mark a straight line. Why?

But it may require three letters to designate a curve. Why?

DIVISION OF SUBJECT.

63. By combinations of lines upon a plane, PLANE FIGURES are formed, which may or may not inclose an area.

By combinations of lines and surfaces, figures are formed in space, which may or may not inclose a volume.

In an elementary work, only a few of the infinite variety of geometrical figures that exist, are mentioned, and only the leading principles concerning those few.

Elementary Geometry is divided into PLANE GEOMETRY, which treats of plane figures, and GEOMETRY IN SPACE, which treats of figures whose points are not all in one plane.

In Plane Geometry, we will first consider lines without reference to area, and afterward inclosed figures.

In Geometry in Space, we will first consider lines and surfaces which do not inclose a space; and afterward, the properties of certain solids.

PLANE GEOMETRY.

CHAPTER III.

STRAIGHT LINES.

64. Problem.—*Straight lines may be added together, and one straight line may be subtracted from another.*

For a straight line may be produced to any extent. Therefore, the length of a straight line may be increased by the length of another line, or two lines may be added together, or we may find the sum of several lines (35).

Again, any straight line may be applied to another, and the two will coincide to their mutual extent. One line may be subtracted from another, by applying the less to the greater and noting the difference.

65. Problem.—*A straight line may be multiplied by any number.*

For several equal lines may be added together.

66. Problem.—*A straight line may be divided by another.*

By repeating the process of subtraction.

67. Problem.—*A straight line may be decreased in any ratio, or it may be divided into several equal parts.*

This is a corollary of the postulate of extent (35).

PROBLEMS IN DRAWING.

68. Exercises in linear drawing afford the best applications of the principles of geometry. Certain lines or combinations of lines being given, it is required to construct other lines which shall have certain geometrical relations to the former.

Except the paper and pencil, or blackboard and crayon, the only instruments used are the ruler and compasses; and all the required lines must be drawn by the aid of these only. The reason for this rule will be shown in the following chapter.

The ruler must have one edge straight. The compasses have two legs with pointed ends, which meet when the instrument is shut. For blackboard work, a stretched cord may be substituted for the compasses.

69. With the *ruler*, a straight line may be drawn on any plane surface, by placing the ruler on the surface and drawing the pencil along the straight edge.

A straight line may be drawn through any two points, after placing the straight edge in contact with the points.

A terminated straight line may be produced after applying the straight edge to a part of it, in order to fix the direction.

70. With the *compasses*, the length of a given line may be taken by opening the legs till the fine points are one on each end of the line. Then this length may be measured on the greater line as often as it will contain the less. A line may thus be produced any required length.

71. The student must distinguish between the problems of geometry and problems in drawing. The former state what can be done with pure geometrical magnitudes, and their truth depends upon showing that they are not incompatible with the nature of the given figure; for a geometrical figure can have any conceivable form or extent.

The problems in drawing corresponding to those above given, except the last, "to divide a given straight line into proportional or equal parts," are solved by the methods just described.

72. The complete discussion of a problem in drawing includes, besides the demonstration and solution, the showing whether the problem has only one solution or several, and the conditions of each.

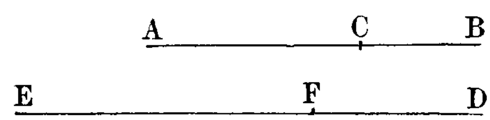
STRAIGHT LINES SIMILAR.

73. Theorem.—*Any two straight lines are similar figures.*

For each has one invariable direction. Hence, two straight lines have the same form, and can differ from each other only in their extent (37).

74. Any straight line may be diminished in any ratio (67), and may therefore be divided in any ratio.

The points in two lines which divide them in the same ratio are homologous points, by the definition (37). Thus, if the lines AB and ED are divided at the points C and F, so that AC : CB :: EF : FD,



then C and F are homologous, or similarly situated points in these lines; AC and EF are homologous parts, and CB and FD are homologous parts.

75. Corollary.—Two homologous parts of two straight lines have the same ratio as the two whole lines.

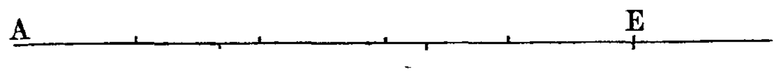
For, AC+CB : EF+FD :: AC : EF (23).
That is, AB : ED :: AC : EF.
Also, AB : ED :: CB : FD.

76. Problem in Drawing.—*To find the ratio of two given straight lines.*

Take, for example, the lines b and c.

If these two lines have a common multiple, that is, a line which contains each of them an exact number of times, let x be the number of times that b is contained in the least common multiple of the two lines, and y the number of times it contains c. Then x times b is equal to y times c.

Therefore, from a point A, draw an indefinite straight line AE.



Apply each of the given lines to it a number of times in succession. The ends of the two lines will coincide after x applications of b, and y applications of c.

If the ends coincide for the first time at E, then AE is the least common multiple of the two lines.

The values of x and y may be found by counting, and these express the ratio of the two lines. For since y times c is equal to x times b, it follows that $b : c :: y : x$, which in this case is as 3 to 5.

It may happen that the two lines have no common multiple. In that case the ends will never exactly coincide after any number of applications to the indefinite line; and the ratio can not be exactly expressed by the common numerals.

By this method, however, the ratio may be found within any desired degree of approximation.

77. But this means is liable to all the sources of error that arise from frequent measurements. In practice, it is usual to measure each line as nearly as may be with a comparatively small standard. The numbers thus found express the ratio nearly.

Whenever two lines have any geometrical dependence upon each other, the ratio may be found by calculation with an accuracy which no measurement by the hand can reach.

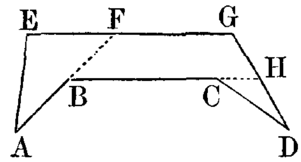
BROKEN LINES.

78. A curve or a broken line is said to be CONCAVE on the side toward the straight line which joins two of its points, and CONVEX to the other side.

79. Theorem.—*A broken line which is convex toward another line that unites its extreme points, is shorter than that line.*

The line ABCD is shorter than the line AEGD, toward which it is convex.

Produce AB and BC till they meet the outer line in F and H.



Since CD is shorter than CHD, it follows (8) that the line ABCD is shorter than ABHD. For a similar reason, ABHD is shorter than AFGD, and AFGD is shorter than AEGD. Therefore, ABCD is shorter than AEGD.

The demonstration would be the same if the outer line were curved, or if it were partly convex to the inner line.

EXERCISE.

80. Vary the above demonstration by producing the lines DC and CB to the left, instead of AB and BC to the right, as in the text; also,

By substituting a curve for the outer line; also,

By letting the inner line consist of two or of four straight lines.

81. A fine thread being tightly stretched, and thus forced to assume that position which is the shortest path between its ends, is a good representation of a straight line. Hence, a stretched cord is used for marking straight lines.

The word *straight* is derived from "*stretch*," of which it is an obsolete participle.

ANGLES.

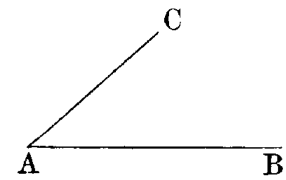
82. An ANGLE is the difference in direction of two lines which have a common point.

83. Theorem.—*The two lines which form an angle lie in one plane, and determine its position.*

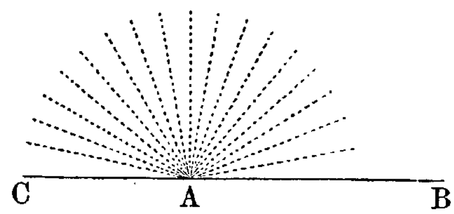
For the plane may pass through the common point and another point in each line, making three in all. These three points determine the position of the plane (60).

DEFINITIONS.

84. Let the line AB be fixed, and the line AC revolve in a plane about the point A; thus taking every direction from A in the plane of its revolution. The angle or difference in direction of the two lines will increase from zero, when AC coincides with AB, till AC takes the direction exactly opposite that of AB.



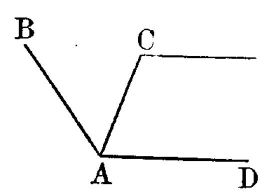
If the motion be continued, AC will, after a complete revolution, again coincide with AB.



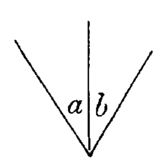
The lines which form an angle are called the SIDES, and the common point is called the VERTEX.

The definition shows that the angle depends upon the directions only, and not upon the length of the sides.

85. Three letters may be used to mark an angle, the one at the vertex being in the middle, as the angle BAC. When there can be no doubt what angle is intended, one letter may answer, as the angle C.



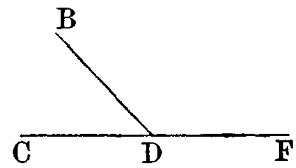
It is frequently convenient to mark angles with letters placed between the sides, as the angles *a* and *b*.



Two angles are ADJACENT when they have the same vertex and one common side between them. Thus, in the last figure, the angles *a* and *b* are adjacent; and, in the previous figure, the angles BAC and CAD.

86. A straight line may be regarded as generated by a point from either end of it, and therefore every straight line has two directions, which are the opposite of each other. We speak of the direction from A to B as the direction AB, and of the direction from B to A as the direction BA.

One line meeting another at some other point than the extremity, makes two angles with it. Thus the angle BDF is the difference in the directions DB and DF; and the angle BDC is the difference in the directions DB and DC.



When two lines pass through or cut each other, four angles are formed, each direction of one line making a difference with each direction of the other.

The opposite angles formed by two lines cutting each other are called VERTICAL angles.

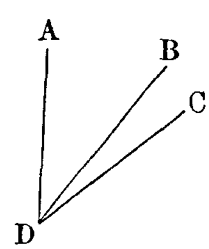
A line which cuts another, or which cuts a figure, is called a SECANT.

PROBLEMS ON ANGLES.

87. Angles may be compared by placing one upon the other, when, if they coincide, they are equal.

Problem.—*One angle may be added to another.*

Let the angles ADB and BDC be adjacent and in the same plane. The angle ADC is plainly equal to the sum of the other two (9).



Problem.—*An angle may be subtracted from a greater one.*

For the angle ADB is the difference between ADC and BDC.

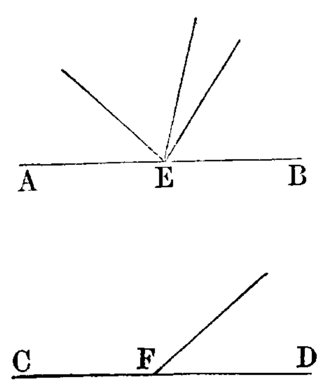
It is equally evident that an angle may be a multiple or a part of another angle; in a word, that angles are quantities which may be compared, added, subtracted, multiplied, or divided.

But angles are not magnitudes, for they have no extent, either linear, superficial, or solid.

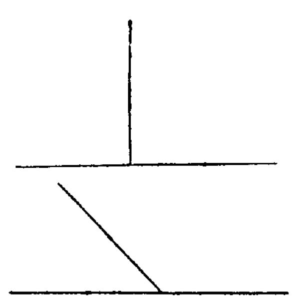
ANGLES FORMED AT ONE POINT.

88. Theorem.—*The sum of all the successive angles formed in a plane upon one side of a straight line, is an invariable quantity; that is, all such sums are equal to each other.*

If AB and CD be two straight lines, then the sum of all the successive angles at E is equal to the sum of all those at F.



For the line AE may be placed on CF, the point E on the point F. Then EB will fall on FD, for when two straight lines coincide in part, they must coincide throughout their mutual extent (51). Therefore, the sum of all the angles upon AB exactly coincides with the sum of all the angles upon CD, and the two sums are equal.



89. When one line meets another, making the adjacent angles equal, the angles are called RIGHT ANGLES.

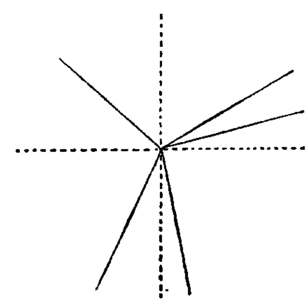
One line is PERPENDICULAR to the other when the angle which they make is a right angle.

Two lines are OBLIQUE to each other when they make an angle which is greater or less than a right angle.

90. Corollary.—All right angles are equal.

For each is half of the sum of the angles upon one side of a straight line. By the above theorem, these sums are always equal, and (7) the halves of equal quantities are equal.

91. Corollary.—The sum of all the successive angles formed in a plane and upon one side of a straight line, is equal to two right angles.



92. Corollary.—The sum of all the successive angles formed in a plane about a point, is equal to four right angles.

93. Corollary.—When two lines cut each other, if one of the angles thus formed is a right angle, the other three must be right angles.

94. In estimating or measuring angles in geometry, the right angle is taken as the standard.

An angle less than a right angle is called ACUTE.

An angle greater than one right angle and less than the sum of two, is called OBTUSE. Angles greater than the sum of two right angles are rarely used in elementary geometry.

When the sum of two angles is equal to a right angle, each is the COMPLEMENT of the other.

When the sum of two angles is equal to two right angles, each is the SUPPLEMENT of the other.

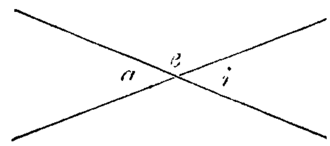
95. Corollary.—Angles which are the complement of the same or of equal angles are equal (7).

96. Corollary.—Angles which are the supplements of the same or of equal angles are equal.

97. Corollary.—The supplement of an obtuse angle is acute.

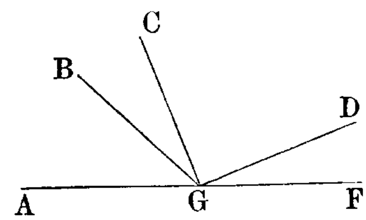
98. Corollary.—The greater an angle, the less is its supplement.

99. Corollary.—Vertical angles are equal. Thus, *a* and *i* are each supplements of *e*.



100. Theorem.—*When the sum of several angles in a plane having their vertices at one point is equal to two right angles, the extreme sides form one straight line.*

If the sum of AGB, BGC, etc., be equal to two right angles, then will AGF be one straight line.



For the sum of all these angles being equal (91) to the sum of the angles upon one side of a straight line, it follows that the two sums may coincide (40), or that AGF may coincide with a straight line. Therefore, AGF is a straight line.

EXERCISES.

101. Which is the greater angle, *a* or *b*, and why?

What is the greatest number of points in which two straight lines may cut each other? In which three may cut each other? Four?

102. The student should ask and answer the question "why" at each step of every demonstration; also, for every corollary. Thus:

Why are vertical angles equal? Why are supplements of the same angles equal?

And in the last theorem: Why is AGF a straight line? Why may AGF coincide with a straight line? Why may the two sums named coincide? Why are the two sums of angles equal?

PERPENDICULAR AND OBLIQUE LINES.

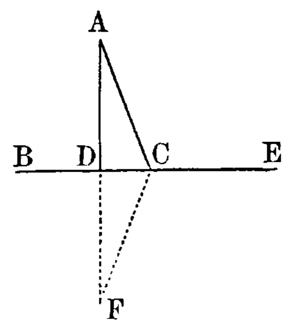
103. Theorem.—*There can be only one line through a given point perpendicular to a given straight line.*

For, since all right angles are equal (90), all lines lying in one plane and perpendicular to a given line, must have the same direction. Now, through a given point in one direction there can be only one straight line (49).

Therefore, since the perpendiculars have the same direction, there can be through a given point only one perpendicular to a given straight line.

When the point is *in* the given line, this theorem must be limited to one plane.

104. Theorem.—*If a perpendicular and oblique lines fall from the same point upon a given straight line, the perpendicular is shorter than any oblique line.*



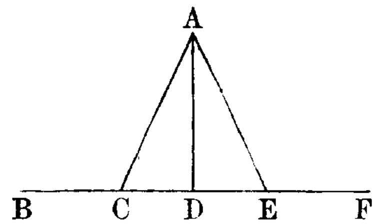
If AD is perpendicular and AC oblique to BE, then AD is shorter than AC.

Let the figure revolve upon BE as upon an axis (61), the point A falling upon F, and the lines AD and AC upon FD and FC.

Now, the angle CDF is equal to the angle CDA, and both are right angles. Therefore, the sum of those two angles being equal to two right angles (100), ADF is a straight line, and is shorter than ACF (54). Therefore, AD, the half of ADF, is shorter than AC, the half of ACF.

105. Corollary.—The distance from a point to a straight line is the perpendicular let fall from the point to the line.

106. Theorem.—*If a perpendicular and several oblique lines fall from the same point upon a given straight line, and if two oblique lines meet the given line at equal distances from the foot of the perpendicular, the two are equal.*



Let AD be the perpendicular and AC and AE the oblique lines, making CD equal to DE. Then AC and AE are equal.

Let that portion of the figure on the left of AD turn upon AD. Since the angles ADB and ADF are equal, DB will take the direction DF; and since DC and DE are equal, the point C will fall on E. Therefore, AC and AE will coincide (51), and are equal.

107. Corollary.—When the oblique lines are equal, the angles which they make with the perpendicular are equal. For CAD may coincide with DAE.

108. Theorem.—*If a line be perpendicular to another at its center, then every point of the perpendicular is equally distant from the two ends of the other line.*

For straight lines extending from any point of the perpendicular to the two ends of the other line must be equal (106).

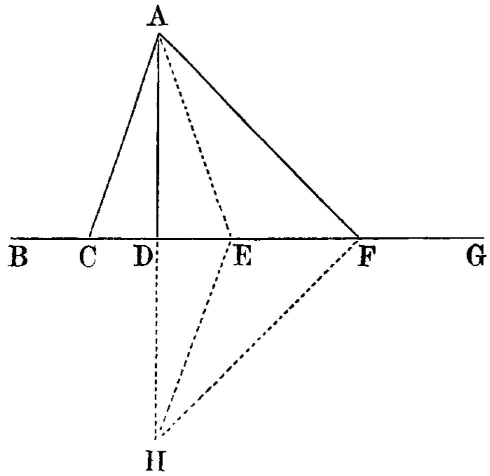
Let the student make a diagram of this. Then state what lines are given by the hypothesis, and what are constructed for demonstration.

109. Corollary.—Since two points fix the position of a line, if a line have two points each equidistant from the ends of another line, the two lines are perpendicular to each other, and the second line is bisected.

The two points may be on the same side, or on opposite sides of the second line.

110. Theorem.—*If a perpendicular and several oblique lines fall from the same point on a given straight line, of two oblique lines, that which meets the given line at a greater distance from the perpendicular is the longer.*

If AD be perpendicular to BG, and DF is greater than DC, then AF is greater than AC.



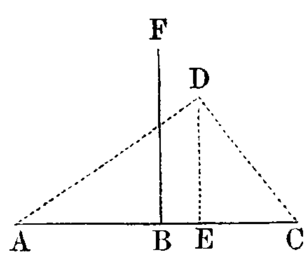
On the line DF take a part DE equal to DC, and join AE. Then let the figure revolve upon BG, the point A falling upon H, and the lines AD, AE, and AF upon HD, HE, and HF.

Now, AEH is shorter than AFH (79); therefore, AE, the half of AEH, is shorter than AF, the half of AFH. But AC is equal to AE (106). Hence, AF is longer than AC, or AE, or any line from A meeting the given line at a less distance from D than DF.

111. Corollary.—A point may be at the same distance from two points of a straight line, one on each side of the perpendicular; but it can not be at the same distance from more than two points.

112. Theorem.—*If a line be perpendicular to another at its center, every point out of the perpendicular is nearer to that end of the line which is on the same side of the perpendicular.*

If BF is perpendicular to AC at its center B, then D, a point not in BF, is nearer to C than to A.



Join DA and DC, and let the perpendicular DE fall from D upon the line AC.

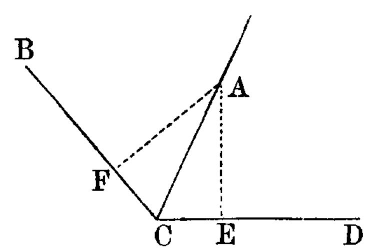
This perpendicular must fall on the same side of BF as the point D, for if it crossed the line BF, there would be from the point of intersection two perpendiculars on AC, which is impossible (103). Now, since AB is equal to BC, AE must be greater than EC. Hence, AD is greater than CD (110).

The point D is supposed to be in the plane of ACD. If it were not, the perpendicular from it might fall on the point B.

BISECTED ANGLE.

113. Theorem.—*Every point of the line which bisects an angle is equidistant from the sides of the angle.*

Let BCD be the given angle, and AC the bisecting line. Then the distance of the two sides from any point A of that line is measured by perpendiculars to the sides, as AF and AE.



Since the angles BCA and DCA are equal, that part of the figure upon the one side of AC may revolve upon AC, and the line BC will take the direction of CD, and coincide with it.

Then the perpendiculars AF and AE must coincide (103), and the point F fall upon E. Therefore, AF and AE are equal, and the point A is equally distant (105) from the sides of the given angle.

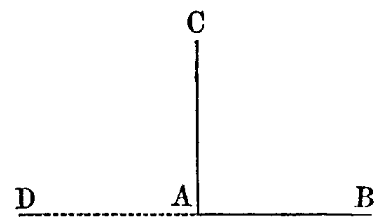
APPLICATION.

114. Perpendicular lines are constantly used in architecture, carpentry, stone-cutting, machinery, etc.

The mason's square consists of two flat rulers made of iron, and connected together in such a manner that both edges of one are at right angles to those of the other. The carpenter's square is much like it, but one of the legs is wood. This instrument is used for drawing perpendicular lines, and for testing the correctness of right angles.

The square itself should be tested in the following manner:

On any plane surface draw an angle, as BAC, with the square. Extend BA in the same straight line to D. Then turn the square so that the edges by which the angle BAC was described, may be applied to the angle DAC. If the coincidence is exact, the square is correct as to these edges.



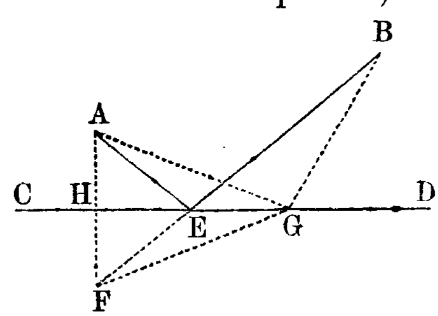
Let the student show that this method of testing the square is according to geometrical principles.

The square here described is not the geometrical figure of that name, which will be defined hereafter.

A MINIMUM LINE.

115. Theorem.—*Of any two lines which may extend from two given points outside of a straight line to any point in it, those are together least which make equal angles with that line.*

Let CD be the line and A and B the points, and AEB the shortest line that can be made from A to B through any point of CD. Then it is to be proved that AEC and BED are equal angles.



Make AH perpendicular to CD, and produce it to F, making HF equal to AH.

Now every point of the line CD is equally distant from A and F (108). Therefore, every line joining B to

F through some point of CD, is equal to a line joining B to A through the same point. Thus, BGF is equal to BGA, since GF and GA are equal. So, BEF is equal to BEA.

But BEA is, by hypothesis, the shortest line from B to A through any point of CD. Therefore, BEF is the shortest line from B to F, and is a straight line (54).

Since BEF is one straight line, the angles FEH and BED are vertical and equal (99). But the angles FEH and AEH are equal (107). Therefore, AEH and BED are equal (6).

116. When several magnitudes are of the same kind but vary in extent, the least is called a *minimum*, and the greatest a *maximum*.

APPLICATION.

When a ray of light is reflected from a polished surface, the incident and reflected parts of the ray make equal angles with the surface. We learn from this geometrical principle that light, when reflected, still adheres to that law of its nature which requires it to take the shortest path.

PARALLELS.

117. PARALLEL lines are straight lines which have the same directions.

118. Corollary.—Two lines which are each parallel to a third are parallel to each other.

119. Corollary.—From the above definition, and the *Axiom of Direction* (49), it follows that there can be only one line through a given point parallel to a given line.

120. Corollary.—From the same premises, it follows that two parallel lines can never meet, or have a common point.

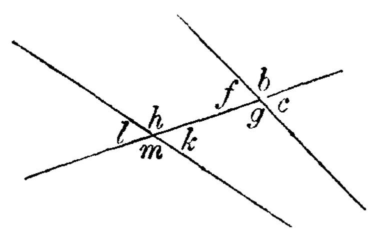
121. Theorem.—*Two parallel lines both lie in one plane and determine its position.*

The position of a plane is determined (60) by either line and one point of the other line. Now the plane has the direction of the first line and can not vary from it (56), and the second line has also the same direction (117) and can not vary from it (44).

Therefore, the second line must also lie wholly in the plane.

NAMES OF ANGLES.

122. When two straight lines are cut by a secant, the eight angles thus formed are named as follows:



The four angles between the two lines are INTERIOR; as, *f*, *g*, *h*, and *k*. The other four are EXTERIOR; as, *b*, *c*, *l*, and *m*.

Two angles on the same side of the secant, and on the same side of the two lines cut by it, are called CORRESPONDING angles. The angles *h* and *b* are corresponding.

Two angles on opposite sides of the secant, and on opposite sides of the two lines cut by it, are called ALTERNATE angles. The angles *f* and *k* are alternate; also, *b* and *m*.

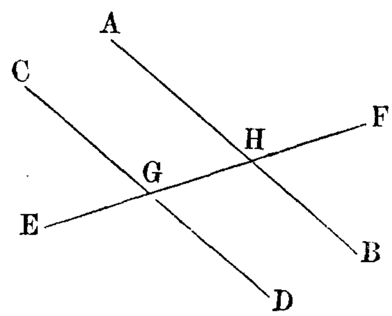
The student should name the corresponding and the alternate angles of each of the eight angles in the above diagram. Let him also name them in the diagram of the following theorem.

123. Corollary.—The corresponding and the alternate angles of any given angle are vertical to each other, and therefore equal (99).

PARALLELS CUT BY A SECANT.

124. Theorem.—*When two parallel lines are cut by a secant, each of the eight angles is equal to its corresponding angle.*

If the straight lines AB and CD have the same directions, then the angles FHB and FGD are equal.



For, since the directions GD and HB are the same, the direction GF differs equally from them. Therefore, the angles are equal (82).

In the same manner, it may be shown that any two corresponding angles are equal.

125. Corollary.—When two parallel lines are cut by a secant, each of the eight angles is equal to its alternate (123).

126. Corollary.—Two interior angles on the same side of the secant are supplements of each other. For, since GHB is the supplement of FHB (91), it is also the supplement of its equal HGD. Two exterior angles on the same side of the secant are supplementary, for a similar reason.

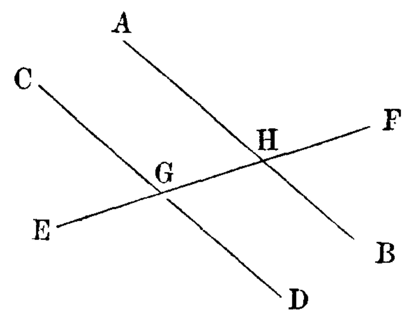
127. Corollary.—When a secant is perpendicular to one of two parallels, it is also perpendicular to the other, and all the angles are right.

Let the student illustrate by a diagram, in this and in all cases when a diagram is not given.

128. Corollary.—When the secant is oblique to the parallels, four of the angles formed are obtuse and are equal to each other; the other four are acute, and equal; and any acute angle is the supplement of any obtuse.

129. Theorem.—*When two straight lines, being in the same plane, are cut by a third, making the corresponding angles equal, the two lines so cut are parallel.*

If AB and CD lie in the same plane, and if the angles AHF and CGF are equal, then AB and CD are parallel.



For, suppose a straight line to pass through the point H, parallel to DC. Such a line makes a corresponding angle equal to CGF, and therefore equal to AHF. This supposed parallel line lies in the same plane as CD and H (121); that is, by hypothesis, in the same plane as AB. But if it lies in the same plane with AB and makes the same angle with the same line EF, at the same point H, then it must coincide with AB. For, when two angles are equal and placed one upon the other, they coincide throughout. Therefore, AB is parallel to CD.

130. Corollary.—If the alternate angles are equal, the lines are parallel (123).

131. Corollary.—The same conclusion must follow when the interior angles on the same side of the secant are supplementary.

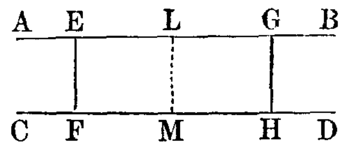
DISTANCE BETWEEN PARALLELS.

132. Theorem.—*Two parallel lines are everywhere equally distant.*

The distance between two parallel lines is measured by a line perpendicular to them, since it is the shortest from one to the other.

Let AB and CD be two parallels. Then any perpendiculars to them, as EF and GH, are equal. From M, the center of FH, erect the perpendicular ML.

Let that part of the figure to the left of ML revolve upon ML. All the angles of the figure being right angles, MC will fall upon MD. Since MF is equal to MH, the point F will fall on H, and the angles at F and H being equal, FE will take the direction HG, and the point E will be on the line HG. But since the angles at L are equal, the point E will also fall on LB, and being on both LB and HG, it must be on G. Therefore, FE and HG coincide and are equal.

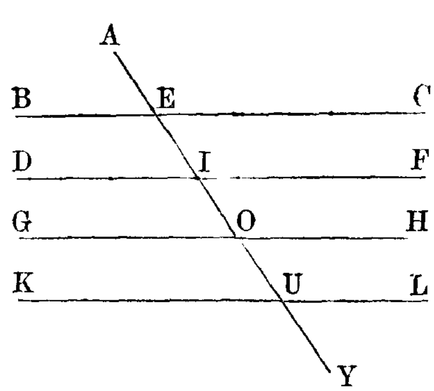


133. Corollary.—The parts of parallel lines included between perpendiculars to them, must be equal. For the perpendiculars are parallel (129).

SECANT AND PARALLELS.

134. Theorem.—*If several equally distant parallel lines be cut by a secant, the secant will be divided into equal parts.*

If the parallels BC, DF, GH, and KL are at equal distances, then the parts EI, IO, and OU of the secant AY are equal.



For that part of the figure included between BC and DF may be placed upon and will coincide with that part between DF and GH; for the parallels are everywhere equally distant (132).

Let them be so placed that the point E may fall upon I. Then, since the angles BEI and DIO (124) are equal, the line EI will take the direction IO. And since DF and GH coincide, the point I will fall on O. Therefore, EI and IO coincide and are equal. In like manner, show that any two of the intercepted parts of the line AY are equal.

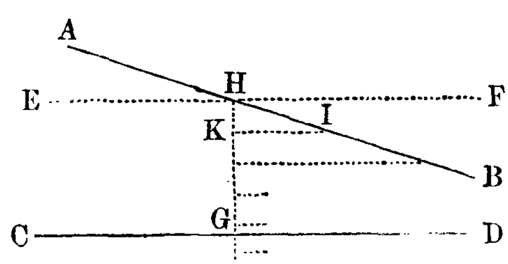
135. Corollary.—Conversely, if several parallel lines intercept equal segments of a secant, then the several distances between the parallels are equal.

136. Corollary.—When the distances between the parallels are unequal, the segments of the secant are unequal. And conversely, when the segments of the secant are unequal, the distances are unequal.

LINES NOT PARALLEL MEET.

137. Theorem.—*If two straight lines are in the same plane and are not parallel, they will meet if sufficiently produced.*

Let AB and CD be two lines. Let the line EF, parallel to CD, pass through any point of AB, as H. From H let the perpendicular HG fall upon CD.



Since AB and EF have different directions, they cut each other at the point H. Take any point, as I, in that part of AB which lies between EF and CD, and extend a line IK parallel to CD through the point I. Now divide HG into parts equal to HK until one of the points of division falls beyond G. Then along HB, take parts equal to HI, as often as

HK was taken along HG. Lastly, from each point of division of HB, extend a line perpendicular to HG.

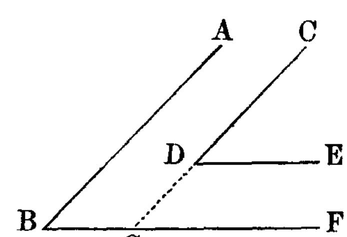
These perpendiculars are parallel to each other and to CD (129). These parallels by construction intercept equal parts of HB. Therefore (135), they are equally distant from each other. Hence, HG is divided by them into equal segments (134); that is, each one passes through one of the previously ascertained points of the line HG.

But the last of these points was beyond the line CD, and as the parallel can not cross CD (120), the corresponding point of HB is beyond CD. Therefore, HB and CD must cross each other.

ANGLES WITH PARALLEL SIDES.

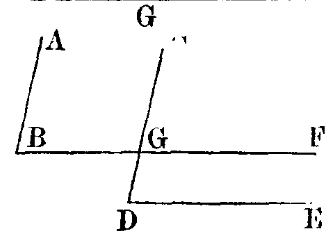
138. Theorem.—*When the sides of one angle are parallel to the sides of another, and have respectively the same directions from their vertices, the two angles are equal.*

If the directions BA and DC are the same, and the directions DE and BF are the same, then the angles ABF and CDE are equal.



For each of these angles is equal to the angle CGF (124).

139. Let the student demonstrate that when two of the parallel sides have opposite directions, and the other two have the same direction, then the angles are supplementary.

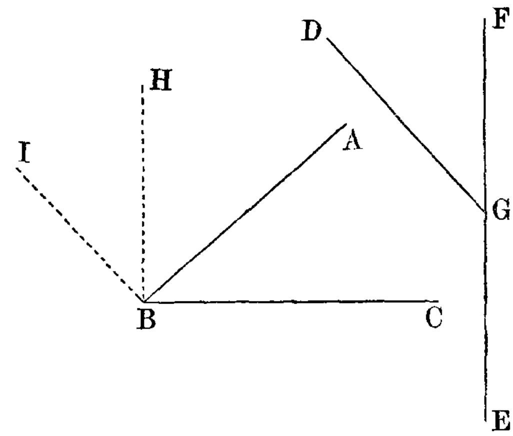


Let him also demonstrate that if both sides of one angle have directions respectively opposite to those of the other, then again the angles are equal.

ANGLES WITH PERPENDICULAR SIDES.

140. Theorem.—*Two angles which have their sides respectively perpendicular are equal or supplementary.*

If AB is perpendicular to DG, and BC is perpendicular to EF, then the angle ABC is equal to one, and supplementary to the other of the angles formed by DG and EF (86).

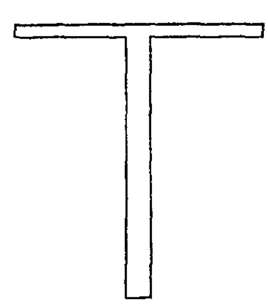


Through B extend BI parallel to GD, and BH parallel to EF.

Now, ABI and CBH are right angles (127), and therefore equal (90). Subtracting the angle HBA from each, the remainders HBI and ABC are equal (7). But HBI is equal to FGD (138), and is the supplement of EGD (139). Therefore, the angle ABC is equal or supplementary to any angle formed by the lines DG and EF.

APPLICATIONS.

141. The instrument called the T square consists of two straight and flat rulers fixed at right angles to each other, as in the figure. It is used to draw parallel lines.



Draw a straight line in a direction perpendicular to that in which it is required to draw parallel lines. Lay the cross-piece of the T ruler along this line. The other piece of the ruler gives the direction of one of the parallels. The ruler being moved along the paper, keeping the cross-piece coincident with the line first described, any number of parallel lines may be drawn.

What is the principle of geometry involved in the use of this instrument?

142. The uniform distance of parallel lines is the principle upon which numerous instruments and processes in the arts are founded.

If two systems, each consisting of several parallel lines, cross each other at right angles, all the parts of one system included between any two lines of the other system will be equal. The ordinary framing of a window consists of two systems of lines of this kind; the shelves and upright standards of book-cases and the paneling of doors also afford similar examples.

143. The joiner's gauge is a tool with which a line may be drawn on a board parallel to its edge. It consists of a square piece of wood, with a sharp steel point near the end of one side, and a movable band, which may be fastened by a screw or key at any required distance from the point. The gauge is held perpendicular to the edge of the board, against which the band is pressed while the tool is moved along the board, the steel point tracing the parallel line.

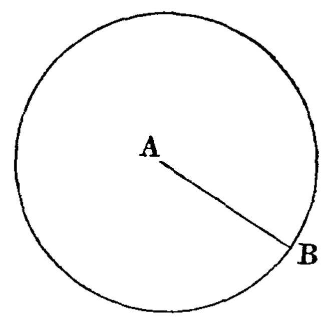
144. It is frequently important in machinery that a body shall have what is called a parallel motion; that is, such that all its parts shall move in parallel lines, preserving the same relative position to each other.

The piston of a steam-engine, and the rod which it drives, receive such a motion; and any deviation from it would be attended with consequences injurious to the machinery. The whole mass of the piston and its rod must be so moved, that every point of it shall describe a line exactly parallel to the direction of the cylinder.

CHAPTER IV.

THE CIRCUMFERENCE.

145. Let the line AB revolve in a plane about the end A, which is fixed. Then the point B will describe a line which returns upon itself, called a circumference of a circle. Hence, the following definitions:



A CIRCLE is a portion of a plane bounded by a line called a CIRCUMFERENCE, every point of which is equally distant from a point within called the CENTER.

146. Theorem.—*A circumference is curved throughout.*

For a straight line can not have more than two points equally distant from a given point (111).

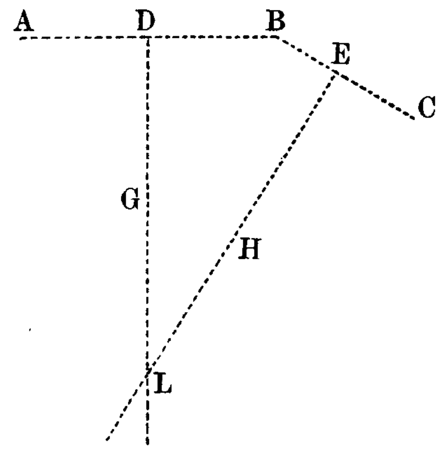
147. Corollary.—A straight line can not cut a circumference in more than two points.

148. The circumference is the only curve considered in elementary geometry. Let us examine the properties of this line, and of the straight lines which may be combined with it.

HOW DETERMINED.

149. Theorem.—*Three points not in the same straight line fix a circumference both as to position and extent.*

The three given points, as A, B, and C, determine the position of a plane. Let the given points be joined by straight lines AB and BC. At D and E, the middle points of these lines, let perpendiculars be erected in the plane of the three points.



By the hypothesis, AB and BC make an angle at B. Therefore, GD is not perpendicular to BC, for if it were, AB and BC would be parallel (129). Hence, DG and EH are not parallel (117), since one is perpendicular and the other is not perpendicular to BC. Therefore, DG and EH will meet (137) if produced. Let L be their point of intersection.

Since every point of DG is equidistant from A and B (108), and since every point of EH is equidistant from B and C, their common point L is equidistant from A, B, and C. Therefore, with this point as a center, a circumference may be described through A, B, and C. There can be no other circumference through these three points, for there is no other point besides L equally distant from all three (112).

Therefore, these three points fix the position and the extent of the circumference which passes through them.

ARCS AND RADII.

150. An Arc is a portion of a circumference.

A Radius is a straight line from the center to the circumference.

A Diameter is a straight line passing through the center, and limited at both ends by the circumference.

A Chord is a straight line joining the ends of an arc.

151. Corollary.—All radii of the same circumference are equal.

152. Corollary.—In the same circumference, a diameter is double the radius, and all diameters are equal.

153. Corollary.—Every point of the plane at greater distance from the center than the length of the radius, is outside of the circumference. Every point at a less distance from the center, is within the circumference. Every point whose distance from the center is equal to the radius, is on the circumference.

154. Theorem.—*Circumferences which have equal radii are equal.*

Let the center of one be placed on that of the other. Then the circumferences will coincide. For if it were otherwise, then some points would be unequally distant from the common center, which is impossible when the radii are equal. Therefore, the circumferences are equal.

155. Corollary.—A circumference may revolve upon, or slide along its equal.

156. Corollary.—Two arcs of the same or of equal circles may coincide so far as both extend.

157. Theorem.—*Every diameter bisects the circumference and the circle.*

For that part upon one side of the diameter may be turned upon that line as its axis. When the two parts thus meet, they will coincide; for if they did not, some points of the circumference would be unequally distant from the center.

158. A line which divides any figure in this manner, is said to divide it *symmetrically;* and a figure which can be so divided is *symmetrical.*

159. Theorem.—*A diameter is greater than any other chord of the same circumference.*

To be demonstrated by the student.

160. Problem.—*Arcs of equal radii may be added together, or one may be subtracted from another.*

For an arc may be produced till it becomes an entire circumference, or it may be diminished at will (35 and 145).

Therefore, the length of an arc may be increased or decreased by the length of another arc of the same radius; and the result, that is, the sum or difference, will be an arc of the same radius.

161. Corollary.—Arcs of equal radii may be multiplied or divided in the same manner as straight lines.

162. The sum of several arcs may be greater than a circumference.

163. Two arcs not having the same radius may be joined together, and the result may be called their sum; but it is not one arc, for it is not a part of one circumference.

APPLICATIONS.

164. The circumference is the only line which can move along itself, around a center, without suffering any change. For any line that can do this must, therefore, have all its points equally distant from the center of revolution; that is, it must be a circumference.

It is in virtue of this property that the axles of wheels, shafts, and other solid bodies which are required to revolve within a hollow mold or casing of their own form, must be circular. If they were of any other form, they would be incapable of revolving without carrying the mold or casing around with them.

165. Wheels which are intended to maintain a carriage always at the same hight above the road on which they roll, must be circular, with the axle in the center.

166. The art of turning consists in the production of the circular form by mechanical means. The substance to be turned is placed in a machine called a lathe, which gives it a rotary motion. The edge of a cutting tool is placed at a distance from the axis of revolution equal to the radius of the intended circle. As the substance revolves, the tool removes every part that is further from the axis than the radius, and thus gives a circular form to what remains.

PROBLEMS IN DRAWING.

167. The compasses enable us to draw a circumference, or an arc of a given radius and given center.

Open the instrument till the points are on the two ends of the given radius. Then fix one point on the given center, and the other point may be made to revolve around in contact with the surface, thus tracing out the circumference.

The revolving leg may have a pen or pencil at the point. In the operation, care should be taken not to vary the opening of the compasses.

168. It is evident that with the ruler and compasses (69),

1. A straight line can be drawn through two given points.
2. A given straight line can be produced any length.
3. A circumference can be described from any center, with any radius.

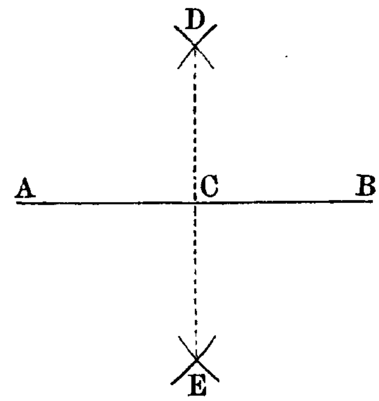
169. The foregoing are the three postulates of Euclid. Since the straight line and the circumference are the only lines treated of in elementary geometry, these Euclidian postulates are a sufficient basis for all problems. Hence, the rule that no instruments shall be used except the ruler and the compasses (68).

170. In the Elements of Euclid, which, for many ages, was the only text-book on elementary geometry, the problems in drawing occupy the place of problems in geometry. At present, the mathematicians of Germany, France, and America put them aside as not forming a necessary part of the theory of the science. English writers, however, generally adhere to Euclid.

171. Problem.—*To bisect a given straight line.*

With A and B as centers, and with a radius greater than the half of AB, describe arcs which intersect in the two points D and E. The straight line joining these two points will bisect AB at C.

Let the demonstration be given by the student (109 and 151).

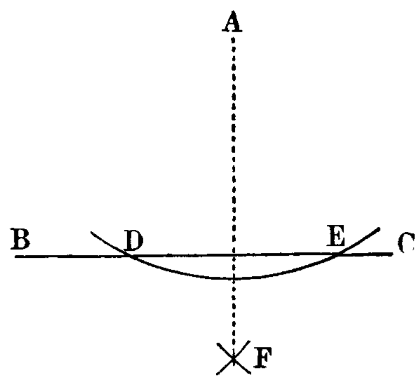


172. Problem.—*To erect a perpendicular on a given straight line at a given point.*

Take two points in the line, one on each side of the given point, at equal distances from it. Describe arcs as in the last problem, and their intersection gives one point of the perpendicular.

Demonstration to be given by the student.

173. Problem.—*To let fall a perpendicular from a given point on a given straight line.*



With the given point as a center, and a radius long enough, describe an arc cutting the given line BC in the points D and E. The line may be produced, if necessary, to be cut by the arc in two places. With D and E as centers, and with a radius greater than the half of DE, describe arcs cutting each other in F. The straight line joining A and F is perpendicular to DE.

Let the student show why.

174. Problem.—*To draw a line through a given point parallel to a given line.*

Let a perpendicular fall from the point on the line. Then, at the given point, erect a perpendicular to this last. It will be parallel to the given line.

Let the student explain why (129).

175. Problem.—*To describe a circumference through three given points.*

The solution of this problem is evident, from Article 149.

176. Problem.—*To find the center of a given arc or circumference.*

Take any three points of the arc, and proceed as in the last problem.

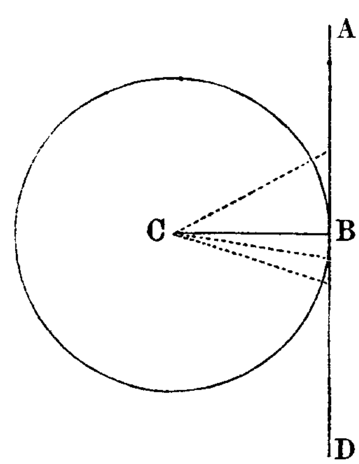
177. The student is advised to make a drawing of every problem. First draw the parts given, then the construction requisite for solution. Afterward demonstrate its correctness.

Endeavor to make the drawing as exact as possible. Let the lines be fine and even, as they better represent the abstract lines of geometry. A geometrical principle is more easily understood by the student, when he makes a neat diagram, than when his drawing is careless.

TANGENT.

178. Theorem.—*A straight line which is perpendicular to a radius at its extremity, touches the circumference in only one point.*

Let AD be perpendicular to the radius BC at its extremity B. Then it is to be proved that AD touches the circumference at B, and at no other point.



If the center C be joined by straight lines with any points of AD, the perpendicular BC will be shorter than any such oblique line (104). Therefore (153), every point of the line AD, except B, is outside of the circumference.

179. A Tangent is a line touching a circumference in only one point. The circumference is also said to be tangent to the straight line. The common point is called the *point of contact.*

APPLICATION.

180. Tangent lines are frequently used in the arts. A common example is when a strap is carried round a part of the circumference of a wheel, and extending to a distance, sufficient tension is given to it to produce such a degree of friction between it and the wheel, that one can not move without the other.

181. Problem in Drawing.—*To draw a tangent at a given point of an arc.*

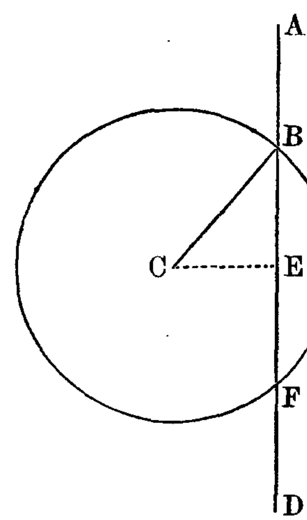
Draw a radius to the given point, and erect a perpendicular to the radius at that point.

It will be necessary to produce the radius beyond the arc, as the student has not yet learned to erect a perpendicular at the extremity of a line without producing it.

SECANT.

182. Theorem.—*A straight line which is oblique to a radius at its extremity, cuts the circumference in two points.*

Let AD be oblique to the radius CB at its extremity B. Then it will cut the circumference at B, and at some other point.



From the center C, let CE fall perpendicularly on AD. On ED, take EF equal to EB.

Then the distance from C to any point of the line AD between B and F is less than the length of the radius CB (110), and to any point of the line beyond B and F, it is greater than the length of CB. Therefore (153), that portion of the line AD between B and F is within, and the parts beyond B and F are without the circumference. Hence, the oblique line cuts the circumference in two points.

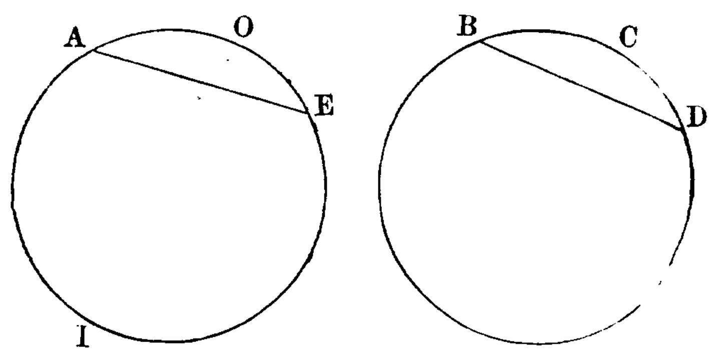
183. Corollary.—A tangent to the circumference is perpendicular to the radius which extends to the point of contact. For, if it were not perpendicular, it would be a secant.

184. Corollary.—At one point of a circumference, there can be only one tangent (103).

CHORDS.

185. Theorem.—*The radii being equal, if two arcs are equal their chords are also equal.*

If the arcs AOE and BCD are equal, and their radii are equal, then AE and BD are equal.



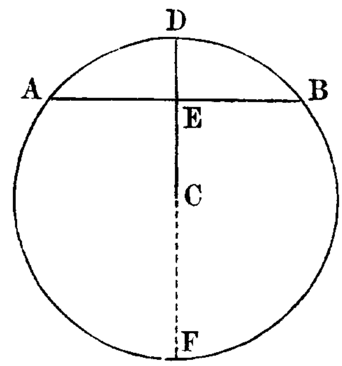
For, since the radii are equal, the circumferences are equal (154); and the arcs may be placed one upon the other, and will coincide, so that A will be upon B, and E upon D. Then the two chords, being straight lines, must coincide (51), and are equal.

186. Every chord subtends two arcs, which together form the whole circumference. Thus the chord AE subtends the arcs AOE and AIE.

The arc of a chord always means the smaller of the two, unless otherwise expressed.

187. Theorem.—*The radius which is perpendicular to a chord bisects the chord and its arc.*

Let CD be perpendicular at E to the chord AB, then will AE be equal to EB, and the arc AD to the arc DB.



Produce DC to the circumference at F, and let that part of the figure on one side of DF be turned upon DF as upon an axis. Then the semi-circumference DAF will coincide with DBF (157). Since the angles at E are right, the line EA will take the direction of EB, and the point A will fall on the point B. Therefore, EA and EB will coincide, and are equal; and the same is true of DA and DB, and of FA and FB.

188. Corollary.—Since two conditions determine the position of a straight line (52), if it has any two of the four conditions mentioned in the theorem, it must have the other two. These four conditions are,

1. The line passes through the center of the circle, that is, it is a radius.
2. It passes through the center of the chord.
3. It passes through the center of the arc.
4. It is perpendicular to the chord.

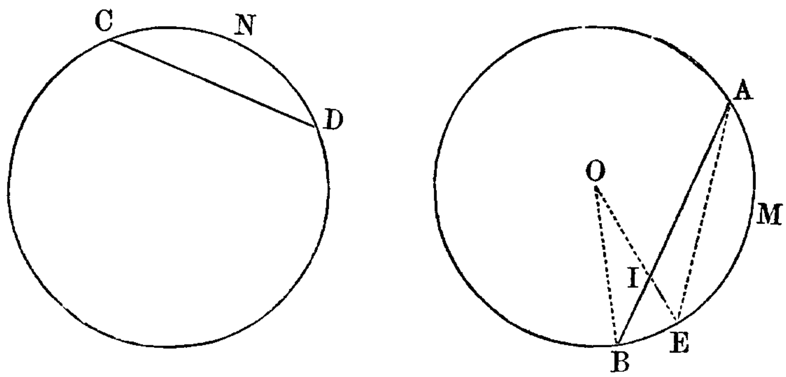
189. Theorem.—*The radii being equal, when two arcs are each less than a semi-circumference, the greater arc has the greater chord.*

If the arc AMB is greater than CND, and the radii of the circles are equal, then AB is greater than CD.

Take AME equal to CND. Join AE, OE, and OB. Then AE is equal to CD (185).

Since the arc AMB is less than a semi-circumference, the chord AB will pass between the arc and the center O, Hence, it cuts the radius OE at some point I.

Now, the broken line OIB is greater than OB (54), or its equal OE. Subtracting OI from each (8), the



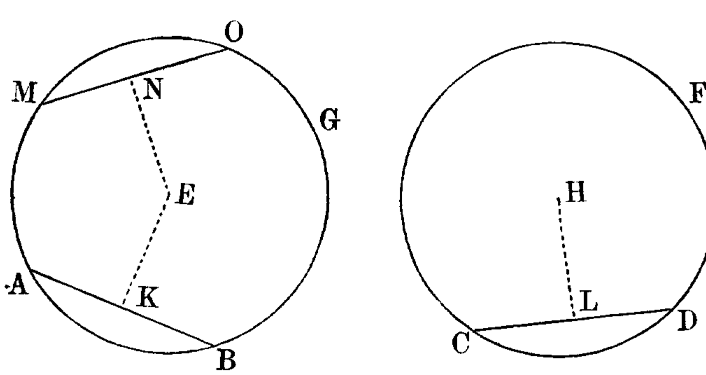
remainder IB is greater than the remainder IE. Adding AI to each of these, we have AB greater than AIE. But AIE is greater than AE. Therefore, AB, the chord of the greater arc, is greater than AE, or its equal CD, the chord of the less.

190. Corollary.—When the arcs are both greater than a semi-circumference, the greater arc has the less chord.

DISTANCE FROM THE CENTER.

191. Theorem.—*When the radii are equal, equal chords are equally distant from the center.*

Let the chords AB and CD be equal, and in the equal



circles ABG and CDF; then the distances of these chords from the centers E and H will also be equal.

Let fall the perpendiculars EK and HL from the centers upon the chords.

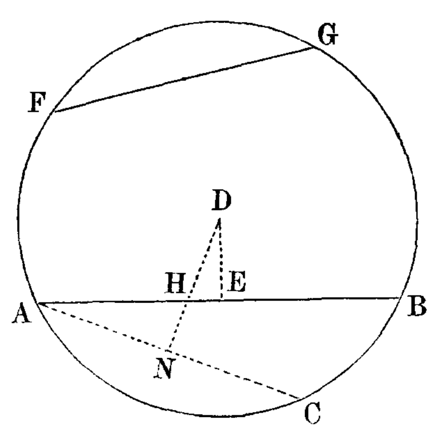
Now, since the chords AB and CD are equal, the arcs AB and CD are also equal (185); and we may apply the circle ABG to its equal CDF, so that they will coincide, and the arc AB coincide with its equal CD. Therefore, the chords will coincide. Since K and L are the middle points of these coinciding chords (187), K will fall upon L. Therefore, the lines EK and HL coincide and are equal. But these equal perpendiculars measure the distance of the chords from the centers (105).

If the equal chords, as MO and AB, are in the same circle, each may be compared with the equal chord CD of the equal circle CDF.

Thus it may be proved that the distances NE and EK are each equal to HL, and therefore equal to each other.

192. Theorem.—*When the radii are equal, the less of two unequal chords is the farther from the center.*

Let AB be the greater of two chords, and FG the less, in the same or an equal circle. Then FG is farther from the center than AB.



Take the arc AC equal to FG. Join AC, and from the center D let fall the perpendiculars DE and DN upon AB and AC.

Since the arc AC is less than AB, the chord AB will be between AC and the center D, and will cut the perpendicular DN. Then DN, the whole, is greater than DH, the part cut off; and DH is greater than DE (104). So much the more is DN greater than DE. Therefore, AC and its equal FG are farther from the center than AB.

193. Corollary.—Conversely of these two theorems, when the radii are equal, chords which are equally distant from the center are equal; and of two chords which are unequally distant from the center, the one nearer to the center is longer than the other.

194. Problem in Drawing.—*To bisect a given arc.*

Draw the chord of the arc, and erect a perpendicular at its center.

State the theorem and the problems in drawing here used.

195. "The most simple case of the division of an arc, after its bisection, is its trisection, or its division into three equal parts. This problem accordingly exercised, at an early epoch in the progress of geometrical science, the ingenuity of mathematicians, and has become memorable in the history of geometrical discovery, for having baffled the skill of the most illustrious geometers.

"Its object was to determine means of dividing any given arc into three equal parts, without any other instruments than the rule and compasses permitted by the postulates prefixed to Euclid's Elements. Simple as the problem appears to be, it never has been solved, and probably never will be, under the above conditions." —*Lardner's Treatise.*

ANGLES AT THE CENTER.

196. Angles which have their vertex at the center of a circle are called, for this reason, *angles at the center.* The arc between the sides of an angle is called the *intercepted arc of the angle.*

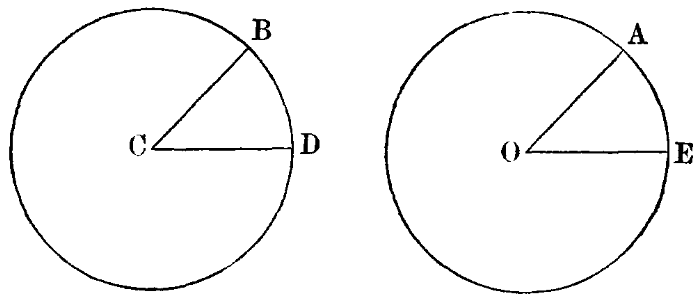
197. Theorem.—*The radii being equal, any two angles at the center have the same ratio as their intercepted arcs.*

This theorem presents the three following cases:

1st. If the arcs are equal, the angles are equal.

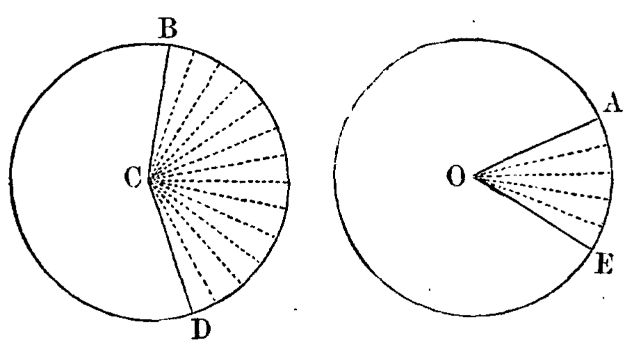
For the arcs may be placed one upon the other, and will coincide. Then BC will coincide with AO, and DC with EO. Thus the angles may coincide, and are equal.

The converse is proved in the same manner.



2d. If the arcs have the ratio of two whole numbers, the angles have the same ratio.

Suppose, for example, the arc BD : arc AE : : 13 : 5.



Then, if the arc BD be divided into thirteen equal parts, and the arc AE into five equal parts, these small arcs will all be equal. Let radii join to their respective centers all the points of division.

The small angles at the center thus formed are all equal, because their intercepted arcs are equal. But BCD is the sum of thirteen, and AOE of five of these equal angles. Therefore,

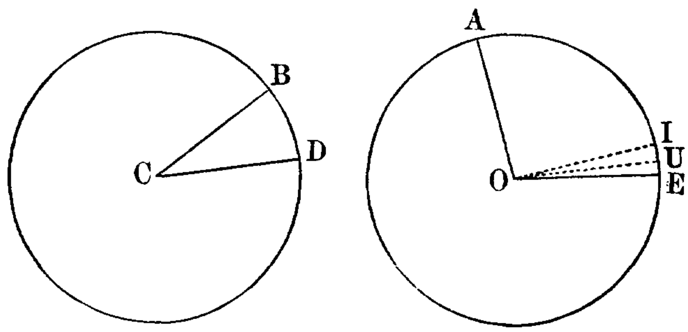
angle BCD : angle AOE : : 13 : 5;

that is, the angles have the same ratio as the arcs.

3d. It remains to be proved, that, if the ratio of the arcs can not be expressed by two whole numbers, the angles have still the same ratio as the arcs; or, that the radius being the same, the

arc BD : arc AE : : angle BCD : angle AOE.

If this proportion is not true, then the first, second,



and third terms being unchanged, the fourth term is either too large or too small. We will prove that it is neither. If it were too large, then some smaller angle, as AOI, would verify the proportion, and

arc BD : arc AE : : angle BCD : angle AOI.

Let the arc BD be divided into equal parts, so small that each of them shall be less than EI. Let one of these parts be applied to the arc AE, beginning at A, and marking the points of division. One of those points must necessarily fall between I and E, say at the point U. Join OU.

Now, by this construction, the arcs BD and AU have the ratio of two whole numbers. Therefore,

arc BD : arc AU : : angle BCD : angle AOU.

These last two proportions may be written thus (19):

arc BD : angle BCD : : arc AE : angle AOI;

arc BD : angle BCD : : arc AU : angle AOU.

Therefore (21),

arc AE : angle AOI : : arc AU : angle AOU;

or (19),

arc AE : arc AU : : angle AOI : angle AOU.

But this last proportion is impossible, for the first antecedent is greater than its consequent, while the second antecedent is less than its consequent. Therefore, the supposition which led to this conclusion is false, and the fourth term of the proportion, first stated, is not too large. It may be shown, in the same way, that it is not too small.

Therefore, the angle AOE is the true fourth term of the proportion, and it is proved that the arc BD is to the arc AE as the angle BCD is to the angle AOE.

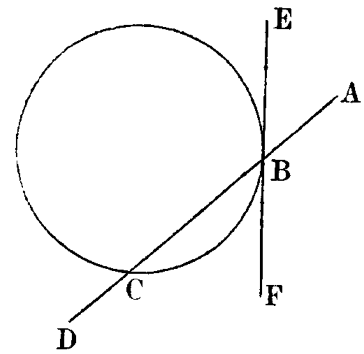
DEMONSTRATION BY LIMITS.

198. The third case of the above proposition may be demonstrated in a different manner, which requires some explanation.

We have this definition of a limit: Let a magnitude vary according to a certain law which causes it to approximate some determinate magnitude. Suppose the first magnitude can, by this law, approach the second indefinitely, but can never quite reach it. Then the second, or invariable magnitude, is said to be the *limit* of the first, or variable one.

199. Any curve may be treated as a limit. The straight parts of a broken line, having all its vertices in the curve, may be diminished at will, and the broken line made to approximate the curve indefinitely. Hence, a curve is the limit of those broken lines which have all their vertices in the curve.

200. The arc BC, which is cut off by the secant AD, may be diminished by successive bisections, keeping the remainders next to B. Thus AD, revolving on the point B, may approach indefinitely the tangent EF. Hence, the tangent at any point of a curve is the limit of the secants which may cut the curve at that point.

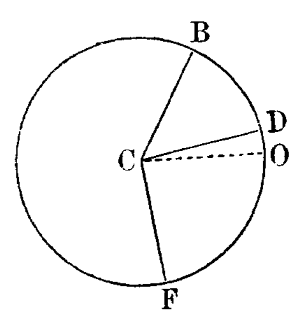


201. The principle upon which all reasoning by the method of limits is governed, is that, *whatever is true up to the limit is true at the limit.* We admit this as an axiom of reasoning, because we can not conceive it to be otherwise.

Whatever is true of every broken line having its vertices in a curve, is true of that curve also. Whatever is true of every secant passing through a point of a curve, is true of the tangent at that point.

We do not say that the arc is a broken line, nor that the tangent is a secant, nor that an arc can be without extent; but that the curve and the tangent are limits toward which variable magnitudes may tend, and that whatever is true all the way to within the least possible distance of a certain point, is true at that point.

202. Having proved (first and second parts, 197) that, when two arcs have the ratio of two whole numbers, the angles at the center have the same ratio, we may then suppose that the ratio of BD to BF can not be expressed by whole numbers.



Now, if we divide BF into two equal parts, the point of division will be at a certain distance from D. We may conceive the arc BF to be divided into any number of equal parts, and by increasing this number, the point O, the point of division nearest to D, may be made to approach within any conceivable distance of D. By the second part of the theorem (197), it is proved that

arc BO : arc BF : : angle BCO : angle BCF.

Now, although the arc BD is itself incommensurable with BF, yet it is the limit of the arcs BO, and the angle BCD is the limit of the angles BCO. Therefore, since whatever is true up to the limit is true at the limit,

arc BD : arc BF : : angle BCD : angle BCF.

That is, the intercepted arcs have the same ratio as their angles at the center.

METHOD OF INFINITES.

203. Modern geometers have made much use of a kind of reasoning which may be called *the method of infinites.* It consists in supposing that any line of definite extent and form is composed of an infinite number of infinitely small straight lines.

A surface is supposed to consist of an infinite number of infinitely narrow surfaces, and a solid of an infinite number of infinitely thin solids. These thin solids, narrow surfaces, and small lines, are called *infinitesimals.*

204. The reasoning of the method of infinites is substantially the same in its logical rigor as of the method of limits. The method of infinites is a much abbreviated form of the method of limits.

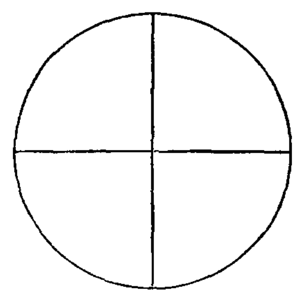
The student must be careful how he adopts it. For when the infinite is brought into an argument by the unskillful, the conclusion is very apt to be absurd. It is sufficient to say, that where the method of limits can be used, the method of infinites may also be used without error.

The method of infinites has also been called the *method of indivisibles.* Some examples of its use will be given in the course of the work.

ARCS AND ANGLES.

We return to the subject of angles at the center. The theorem last given (197) has the following

205. Corollary.—If two diameters are perpendicular to each other, they divide the whole circumference into four equal parts.



206. A QUADRANT is the fourth part of a circumference.

207. Since the angle at the center varies as the intercepted arc, mathematicians have adopted the same method of measuring both angles and arcs. As a right angle is the unit of angles, so a quadrant of a certain radius may be taken as the standard for the measurement of arcs that have the same radius.

For the same reason, we usually say that the intercepted arc measures the angle at the center. Thus, the right angle is said to be measured by the quadrant; half a right angle, by one-eighth of a circumference; and so on.

APPLICATIONS.

208. In the applications of geometry to practical purposes, the quadrant and the right angle are divided into ninety equal parts, each of which is called a degree. Each degree is marked thus °, and is divided into sixty minutes, marked thus ′; and each minute is divided into sixty seconds, marked thus ″.

Hence, it appears that there are in an entire circumference, or in the sum of all the successive angles about a point, 360°, or 21 600′, or 1 296 000″. Some astronomers, mostly the French, divide the right angle and the quadrant into one hundred parts, each of these into one hundred; and so on.

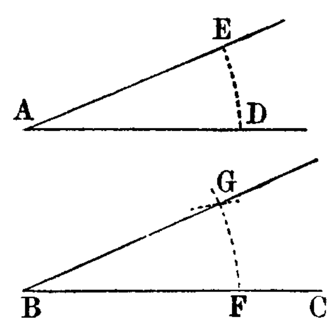
209. Instruments for measuring angles are founded upon the principle that arcs are proportional to angles. Such instruments usually consist either of a part or an entire circle of metal, on the surface of which is accurately engraven its divisions into degrees, etc. Many kinds of instruments used by surveyors, navigators, and astronomers, are constructed upon this principle.

210. An instrument called a protractor is used, in drawing, for measuring angles, and for laying down, on paper, angles of any required size. It consists of a semicircle of brass or mica, the circumference of which is divided into degrees and parts of a degree.

PROBLEMS IN DRAWING.

211. Problem.—*To draw an angle equal to a given angle.*

Let it be required to draw a line making, with the given line BC, an angle at B equal to the given angle A.



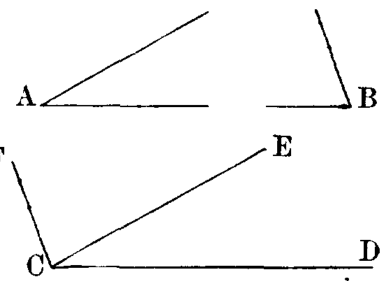
With A as a center, and any assumed radius AD, draw the arc DE cutting the sides of the angle A. With B as a center, and the same radius as before, draw an arc FG. Join DE. With F as a center, and a radius equal to DE, draw an arc cutting FG at the point G. Join BG. Then GBF is the required angle.

For, joining FG, the arcs DE and FG have equal radii and equal chords, and therefore are equal (185). Hence, they subtend equal angles (197).

212. Corollary.—An arc equal to a given arc may be drawn in the same way.

213. Problem.—*To draw an angle equal to the sum of two given angles.*

Let A and B be the given angles. First, make the angle DCE equal to A, and then at C, on the line CE, draw the angle ECF equal to B. The angle FCD is equal to the sum of A and B (9).



214. Corollary.—In a similar manner, draw an angle equal to the sum of several given angles; also, an angle equal to the difference of two given angles; or, an angle equal to the supplement, or to the complement of a given angle.

215. Corollary.—By the same methods, an arc may be drawn equal to the difference of two arcs having equal radii, or equal to the sum of several arcs.

216. Problem.—*To erect a perpendicular to a given line at its extreme point, without producing the given line.*

A right angle may be made separately, and then, at the end of the given line, an angle be made equal to the given angle.

This is the method universally employed by mechanics and draughtsmen to construct right angles and perpendiculars by the use of the *square.*

217. Problem.—*To draw a line through a given point parallel to a given line.*

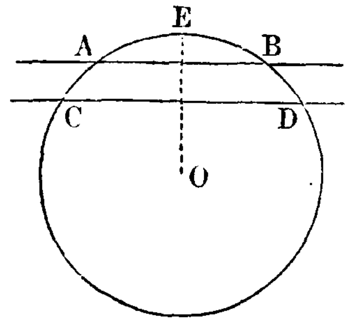
This has been done by means of perpendiculars (174). It may be done with an oblique secant, by making the alternate or the corresponding angles equal.

ARCS INTERCEPTED BY PARALLELS.

218. An arc which is included between two parallel lines, or between the sides of an angle, is called *intercepted.*

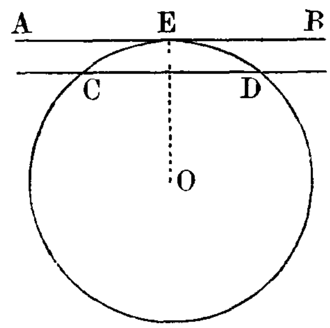
219. Theorem.—*Two parallel lines intercept equal arcs of a circumference.*

The two lines may be both secants, or both tangents, or one a secant and one a tangent.



1st. When both are secants. The arcs AC and BD intercepted by the parallels AB and CD are equal.

For, let fall from the center O a perpendicular upon CD, and produce it to the circumference at E. Then OE is also perpendicular to AB (127). Therefore, the arcs EA and EB are equal (187); and the arcs EC and ED are equal. Subtracting the first from the second, there remains the arc AC equal to the arc BD.

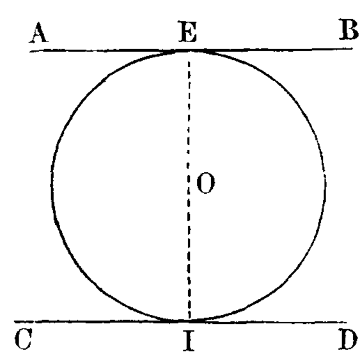


2d. When one is a tangent.

Extend the radius OE to the point of contact. This radius is perpendicular to the tangent AB (183). Hence, it is perpendicular to the secant CD (127), and therefore it bisects the arc CED at the point E (187). That is, the intercepted arcs EC and ED are equal.

3d. When both are tangents.

Extend the radii OE and OI to the points of contact.

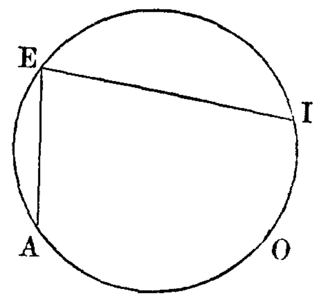


These radii being perpendicular (183) to the parallels, must (103 and 127) form one straight line. Therefore, EI is a diameter, and divides (157) the circumference into equal parts. But these equal parts are the arcs intercepted by the parallel tangents.

Therefore, in every case, the arcs intercepted by two parallels are equal.

ARCS INTERCEPTED BY ANGLES.

220. An Inscribed Angle is one whose sides are chords or secants, and whose vertex is on the circumference. An angle is said to be *inscribed in an arc*, when its vertex is on the arc and its sides extend to or through the ends of the arc. In such a case the arc is said *to contain the angle.* Thus, the angle AEI is inscribed in the arc AEI, and the arc AEI contains the angle AEI.



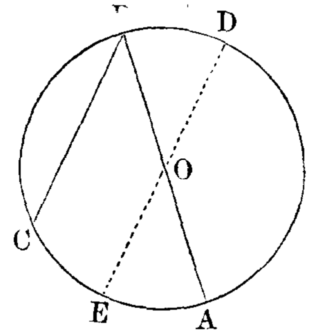
An angle is said to stand upon the arc intercepted between its sides. Thus, the angle AEI stands upon the arc AOI.

221. Corollary.—The arc in which an angle is inscribed, and the arc intercepted between its sides, compose the whole circumference.

222. Theorem.—*An inscribed angle is measured by half of the intercepted arc.*

This demonstration also presents three cases. The center of the circle may be on one of the sides of the angle, or it may be inside, or it may be outside of the angle.

1st. One side of the angle, as AB, may be a diameter.

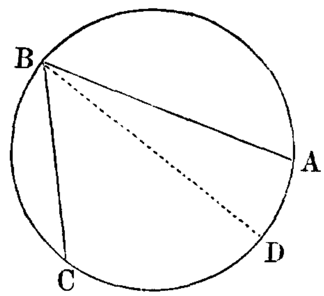


Make the diameter DE, parallel to BC, the other side of the angle. Then the angle B is equal to its alternate angle BOD (125), which is measured by the arc BD (207). This arc is equal to CE (219), and also to EA (197). Therefore, the arc

BD is equal to the half of AC, and the inscribed angle B is measured by half of its intercepted arc.

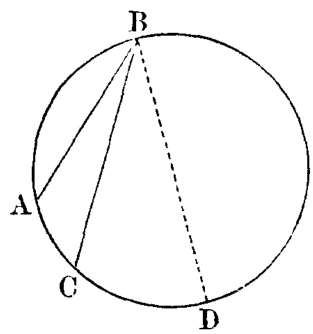
2d. The center of the circle may be within the angle.

From the vertex B extend a diameter to the opposite side of the circumference at D. As just proved, the angle ABD is measured by half of the arc AD, and the angle DBC by half of the arc DC. Therefore, the sum of the two angles, or ABC, is measured by half of the sum of the two arcs, or half of the arc ADC.



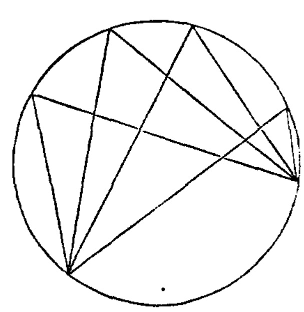
3d. The center of the circle may be outside of the angle.

Extend a diameter from the vertex as before. The angle ABC is equal to ABD diminished by DBC, and is, therefore, measured by half of the arc DA diminished by half of DC; that is, by the half of AC.



223. Corollary.—When an inscribed angle and an angle at the center have the same intercepted arc, the inscribed angle is half of the angle at the center.

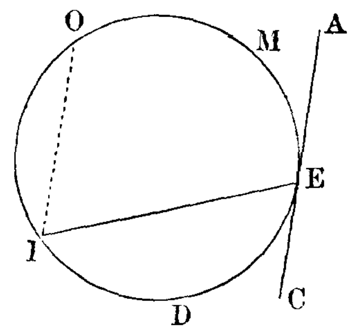
224. Corollary.—All angles inscribed in the same arc are equal, for they have the same measure.



225. Corollary.—Every angle inscribed in a semi-circumference is a right angle. If the arc is less than a semi-circumference, the angle is obtuse. If the arc is greater, the angle is acute.

226. Theorem.—*The angle formed by a tangent and a chord is measured by half the intercepted arc.*

The angle CEI, formed by the tangent AC and the chord EI, is measured by half the intercepted arc IDE.



Through I, make the chord IO parallel to the tangent AC.

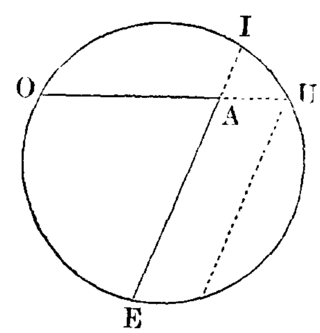
The angle CEI is equal to its alternate EIO (125), which is measured by half the arc OME (222), which is equal to the arc IDE (219). Therefore, the angle CEI is measured by half the arc IDE.

The sum of the angles AEI and CEI is two right angles, and is therefore measured by half the whole circumference (207). Hence, the angle AEI is equal to two right angles diminished by the angle CEI, and is measured by half the whole circumference diminished by half the arc IDE; that is, by half the arc IOME.

Thus it is proved that each of the angles formed at E, is measured by half the arc intercepted between its sides.

227. This theorem may be demonstrated very elegantly by the method of limits (200).

228. Theorem.—*Every angle whose vertex is within the circumference, is measured by half the sum of the arcs intercepted between its sides and its sides produced.*

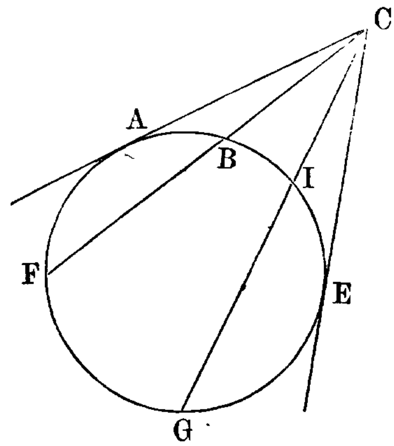


Thus, the angle OAE is measured by half the sum of the arcs OE and IU.

To be demonstrated by the student, using the previous theorems (219 and 222).

229. Theorem.—*Every angle whose vertex is outside of a circumference, and whose sides are either tangent or secant, is measured by half the difference of the intercepted arcs.*

Thus, the angle ACF is measured by half the difference of the arcs AF and AB; the angle FCG, by half the difference of the arcs FG and BI; and the angle ACE, by half the difference of the arcs AFGE and ABIE.

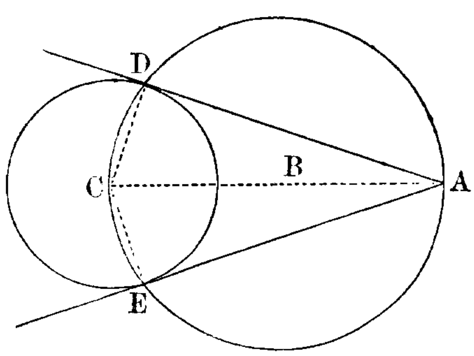


This, also, may be demonstrated by the student, by the aid of the previous theorems on intercepted arcs.

PROBLEMS IN DRAWING.

230. Problem.—*Through a given point out of a circumference, to draw a tangent to the circumference.*

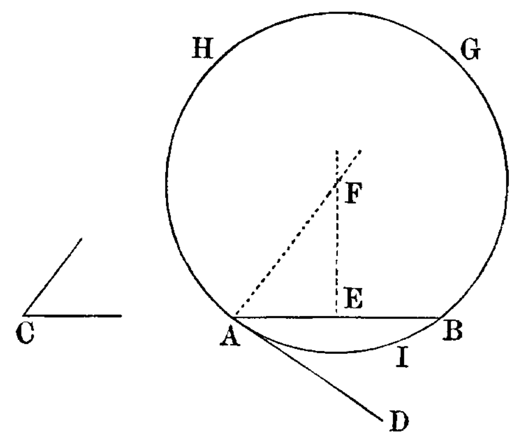
Let A be the given point, and C the center of the given circle. Join AC. Bisect AC at the point B (171). With B as a center and BC as a radius, describe a circumference. It will pass through C and A (153), and will cut the circumference in two points, D and E. Draw straight lines from A through D and E. AD and AE are both tangent to the given circumference.



Join CD and CE. The angle CDA is inscribed in a semicircumference, and is therefore a right angle (225). Since AD is perpendicular to the radius CD, it is tangent to the circumference (178). AE is tangent for the same reasons.

231. Problem.—*Upon a given chord to describe an arc which shall contain a given angle.*

Let AB be the chord, and C the angle. Make the angle DAB equal to C. At A erect a perpendicular to AD, and erect a perpendicular to AB at its center (172). Produce these till they meet at the point F (137). With F as a center, and FA as a radius, describe a circumference. Any angle inscribed in the arc BGHA will be equal to the given angle C.



For AD, being perpendicular to the radius FA, is a tangent (178). Therefore, the angle BAD is measured by half of the arc AIB (226). But any angle contained in the arc AHGB is also measured by half of the same arc (222), and is therefore equal to BAD, which was made equal to C.

POSITIONS OF TWO CIRCUMFERENCES.

232. Theorem.—*Two circumferences can not cut each other in more than two points.*

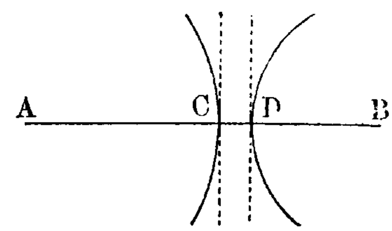
For three points determine the position and extent of a circumference (149). Therefore, if two circumferences have three points common, they must coincide throughout.

233. Let us investigate the various positions which two circumferences may have with reference to each other.

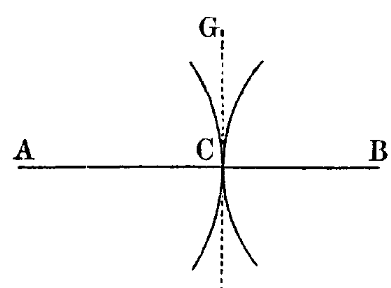
Let A and B be the centers of two circles, and let these points be joined by a straight line, which therefore measures the distance between the centers.

First, suppose the sum of the radii to be less than AB.

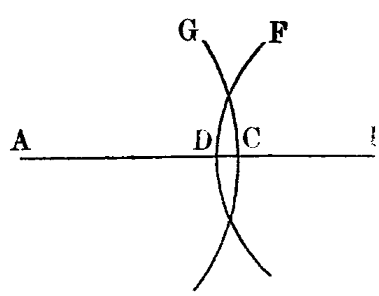
Then AC and BD, the radii, can not reach each other. At C and D, where the curves cut the line AB, let perpendiculars to that line be erected. These perpendiculars are parallel to each other (129). They are also tangent respectively to the two circumferences (178). It follows, therefore, that CD, the distance between these parallels, is also the least distance between the two curves.



234. Next, let the sum of the radii AC and BC be equal to AB, the distance between the centers. Then both curves will pass through the point C (153). At this point let a perpendicular be erected as before. This perpendicular CG is tangent to both the curves (178); that is, it is cut by neither of them. Therefore, the curves have only one common point C.

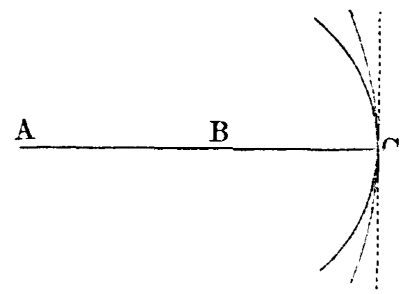


235. Next, let AB be less than the sum, but greater than the difference, of the radii AC and BD. Then the point C will fall within the circumference DF. For if it fell on or outside of it, on the side toward A, then AB would be equal to or greater than the sum of the radii; and if the point C fell on or outside of the curve in the direction toward B, then AB would be equal to or less than the difference between the radii. Each of these is contrary to the hypothesis. For the same reasons, the point D will fall within the



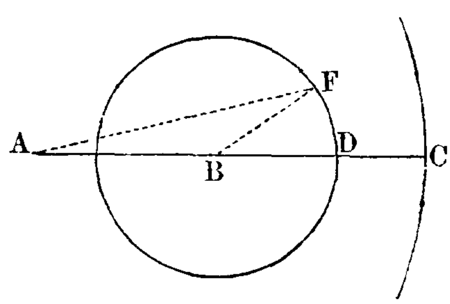
circumference CG. Therefore, these circumferences cut each other, and have two points common (232).

236. Next, let the difference between the two radii AC and BC be equal to the distance AB. A perpendicular to this line at the point C will be a tangent to both curves, and they have a common point at C. They have no other common point, for the two curves are both symmetrical about the line AC (158), and, therefore, if they had a common point on one side of that line, they would have a corresponding common point on the other side; but this can not be, for they would then have three points common (232).



237. Lastly, suppose the distance AB less than the difference of the radii AC and BD, by the line CD. That is,

$$AB + BD + DC = AC.$$



Join A, the center of the larger circle, with F, any point of the smaller circumference, and join BF. Then AB and BD are together equal to AB and BF, which are together greater than AF. Therefore, AD is greater than AF. Hence, the point D is farther from A than any other point of the circumference DF. It follows that CD is the least distance between the two curves.

The above course of reasoning develops the following principles:

238. Theorem.—*Two circumferences may have, with reference to each other, five positions:*

1*st*. *Each may be entirely exterior to the other, when the distance between their centers is greater than the sum of their radii.*

2*d*. *They may touch each other exteriorly, having one point common, when the distance between the centers is equal to the sum of the radii.*

3*d*. *They may cut each other, having two points common, when the distance between the centers is less than the sum and greater than the difference of the radii.*

4*th*. *One may be within the other and tangent, having one point common, when the distance between the centers is equal to the difference of the radii.*

5*th*. *One may be entirely within the other, when the distance between the centers is less than the difference of the radii.*

239. Corollary.—Two circumferences can not have more than one chord common to both.

240. Corollary—The common chord of two circumferences is perpendicular to the straight line which joins their centers and is bisected by it. For the ends of the chords are equidistant from each of the centers, the ends of the other line (109).

241. Corollary.—When two circumferences are tangent to each other, the two centers and the point of contact are in one straight line.

242. Corollary.—When two circumferences have no common point, the least distance between the curves is measured along the line which joins the centers.

243. Corollary.—When the distance between the centers is zero, that is, when they coincide, a straight line through this point may have any direction in the plane; and the two curves are equidistant at all points. Such circles are called CONCENTRIC.

244. A Locus is a line or a surface all the points of which have some common property, which does not belong to any other points. This is also frequently called a *geometrical locus.* Thus,

The circumference of a circle is the locus of all those points in the plane, which are at a given distance from a given point.

A straight line perpendicular to another at its center is the locus of all those points in the plane, which are at the same distance from both ends of the second line.

The geometrical locus of the centers of those circles which have a given radius, and are tangent to a given straight line, is a line parallel to the former, and at a distance from it equal to the radius.

245. The student will find an excellent review of the preceding pages, in demonstrating the theorems, and solving the problems in drawing which follow.

In his efforts to discover the solutions of the more difficult problems in drawing, the student will be much assisted by the following

Suggestions.—1. Suppose the problem solved, and the figure completed.

2. Find the geometrical relations of the different parts of the figure thus formed, drawing auxiliary lines, if necessary.

3. From the principles thus developed, make a rule for the solution of a problem.

This is the *analytic* method of solving problems.

EXERCISES.

1. Take two straight lines at random, and find their ratio.

Make examples in this way for all the problems in drawing.

2. Bisect a quadrant, also its half, its fourth; and so on.

3. From a given point, to draw the shortest line possible to a given straight line.

4. With a given length of radius, to draw a circumference through two given points.

5. From two given points, to draw two equal straight lines which shall end in the same point of a given line.

6. From a point out of a straight line, to draw a second line making a required angle with the first.

7. If from a point without a circle two straight lines extend to the concave part of the circumference, making equal angles with the line joining the same point and the center of the circle, then the parts of the first two lines which are within the circumference are equal.

8. To draw a line through a point such that the perpendiculars upon this line, from two other points, may be equal.

9. From two points on the same side of a straight line, to draw two other straight lines which shall meet in the first, and make equal angles with it.

10. In each of the five cases of the last theorem (238), how many straight lines can be tangent to both circumferences?

The number is different for each case.

11. On any two circumferences, the two points which are at the greatest distance apart are in the prolongation of the line which joins the centers.

12. To draw a circumference with a given radius, through a given point, and tangent to a given straight line.

13. With a given radius, to draw a circumference tangent to two given circumferences.

14. What is the locus of the centers of those circles which have a given radius, and are tangent to a given circle?

15. Of all straight lines which can be drawn from two given points to meet on the convex circumference of a circle, the sum of those two is the least which make equal angles with the tangent to the circle at the point of concourse.

16. If two circumferences be such that the radius of one is the diameter of the other, any straight line extending from their point of contact to the outer circumference is bisected by the inner one.

17. If two circumferences cut each other, and from either point of intersection a diameter be made in each, the extremities of these diameters and the other point of intersection are in the same straight line.

18. If any straight line joining two parallel lines be bisected, any other line through the point of bisection and joining the two parallels, is also bisected at that point.

19. If two circumferences are concentric, a line which is a chord of the one and a tangent of the other, is bisected at the point of contact.

20. If a circle have any number of equal chords, what is the locus of their points of bisection?

21. If any point, not the center, be taken in a diameter of a circle, of all the chords which can pass through that point, that one is the least which is at right angles to the diameter.

22. If from any point there extend two lines tangent to a circumference, the angle contained by the tangents is double the angle contained by the line joining the points of contact and the radius extending to one of them.

23. If from the ends of a diameter perpendiculars be let fall on any line cutting the circumference, the parts intercepted between those perpendiculars and the curve are equal.

24. To draw a circumference with a given radius, so that the sides of a given angle shall be tangents to it.

25. To draw a circumference through two given points, with the center in a given line.

26. Through a given point, to draw a straight line, making equal angles with the two sides of a given angle.

CHAPTER V.

TRIANGLES.

246. Next in regular order is the consideration of those plane figures which inclose an area; and, first, of those whose boundaries are straight lines.

A POLYGON is a portion of a plane bounded by straight lines. The straight lines are the *sides* of the polygon.

The PERIMETER of a polygon is its boundary, or the sum of all the sides. Sometimes this word is used to designate the boundary of any plane figure.

247. A TRIANGLE is a polygon of three sides.

Less than three straight lines can not inclose a surface, for two straight lines can have only one common point (51). Therefore, the triangle is the simplest polygon. From a consideration of its properties, those of all other polygons may be derived.

248. Problem.—*Any three points not in the same straight line may be made the vertices of the three angles of a triangle.*

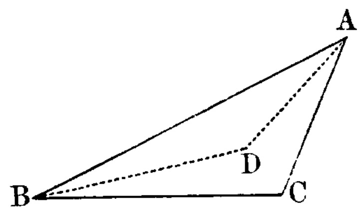
For these points determine the plane (60), and straight lines may join them two and two (47), thus forming the required figure.

INSCRIBED AND CIRCUMSCRIBED.

249. Corollary.—Any three points of a circumference may be made the vertices of a triangle. A circumference may pass through the vertices of any triangle, for it may pass through any three points not in the same straight line (149).

250. Theorem.—*Within every triangle there is a point equally distant from the three sides.*

In the triangle ABC, let lines bisecting the angles A and B be produced until they meet.



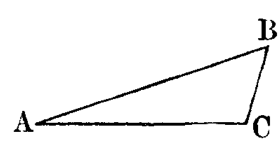
The point D, where the two bisecting lines meet, is equally distant from the two sides AB and BC, since it is a point of the line which bisects the angle B (113). For a similar reason, the point D is equally distant from the two sides AB and AC. Therefore, it is equally distant from the three sides of the triangle.

251. Corollary.—The three lines which bisect the several angles of a triangle meet at one point. For the point D must be in the line which bisects the angle C (113).

252. Corollary.—With D as a center, and a radius equal to the distance of D from either side, a circumference may be described, to which every side of the triangle will be a tangent.

253. When a circumference passes through the vertices of all the angles of a polygon, the circle is said to be *circumscribed* about the polygon, and the polygon to be *inscribed* in the circle. When every side of a polygon is tangent to a circumference, the circle is *inscribed* and the polygon *circumscribed.*

254. The angles at the ends of one side of a triangle are said to be *adjacent* to that side. Thus, the

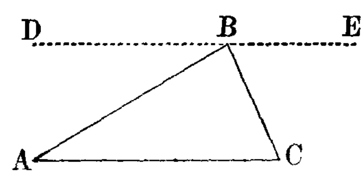


angles A and B are adjacent to the side AB. The angle formed by the other two sides is *opposite*. Thus, the angle A and the side BC are opposite to each other.

SUM OF THE ANGLES.

255. Theorem.—*The sum of the angles of a triangle is equal to two right angles.*

Let the line DE pass through the vertex of one angle, B, parallel to the opposite side, AC.



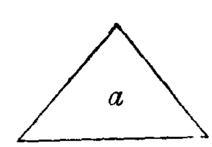
Then the angle A is equal to its alternate angle DBA (125). For the same reason, the angle C is equal to the angle EBC. Hence, the three angles of the triangle are equal to the three consecutive angles at the point B, which are equal to two right angles (91). Therefore, the sum of the three angles of the triangle is equal to two right angles.

256. Corollary.—Each angle of a triangle is the supplement of the sum of the other two.

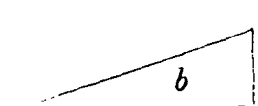
257. Corollary.—At least two of the angles of a triangle are acute.

258. Corollary.—If two angles of a triangle are equal, they are both acute. If the three are equal, they are all acute, and each is two-thirds of a right angle.

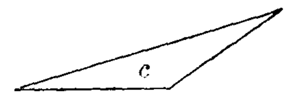
259. An Acute Angled triangle is one which has all its angles acute, as *a*.



A Right Angled triangle has one of the angles right, as *b*.

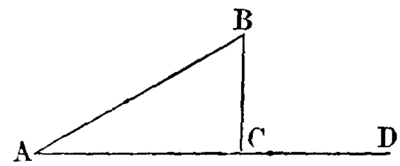


An OBTUSE ANGLED triangle has one of the angles obtuse, as *c*.



260. Corollary.—In a right angled triangle, the two acute angles are complementary (94).

261. Corollary.—If one side of a triangle be produced, the exterior angle thus formed, as BCD, is equal to the sum of the two interior angles not adjacent to it, as A and B (256). So much the more, the exterior angle is greater than either one of the interior angles not adjacent to it.



262. Corollary.—If two angles of a triangle are respectively equal to two angles of another, then the third angles are also equal.

263. Either side of a triangle may be taken as the *base.* Then the vertex of the angle opposite the base is the *vertex* of the triangle.

The ALTITUDE of the triangle is the distance from the vertex to the base, which is measured by a perpendicular let fall on the base produced, if necessary.

264. Corollary.—The altitude of a triangle is equal to the distance between the base and a line through the vertex parallel to the base.

265. When one of the angles at the base is obtuse, the perpendicular falls outside of the triangle.

When one of the angles at the base is right, the altitude coincides with the perpendicular side.

When both the angles at the base are acute, the altitude falls within the triangle.

Let the student give the reason for each case, and illustrate it with a diagram.

LIMITS OF SIDES.

266. Theorem.—*Each side of a triangle is smaller than the sum of the other two, and greater than their difference.*

The first part of this theorem is an immediate consequence of the Axiom of Distance (54); that is,

$$AC < AB + BC.$$

Subtract AB from both members of this inequality, and

$$AC - AB < BC.$$

That is, BC is greater than the difference of the other sides.

Prove the same for each of the other sides.

267. An Equilateral triangle is one which has three sides equal.

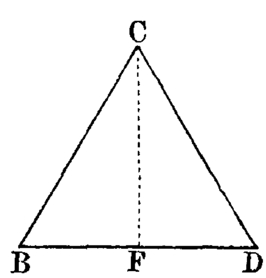
An Isosceles triangle is one which has only two sides equal.

A Scalene triangle is one which has no two sides equal.

EQUAL SIDES.

268. Theorem.—*When two sides of a triangle are equal, the angles opposite to them are equal.*

If the triangle BCD is isosceles, the angles B and D, which are opposite the equal sides, are equal.



Let the angle C be divided into two equal parts, and let the dividing line extend to the opposite side of the triangle at F.

Then, that portion of the figure upon one side of this line may be turned upon it as upon an axis. Since the angle C was bisected, the line BC will fall upon DC; and, since these two lines are equal, the point B will fall upon D. But F, being a point of the axis, remains fixed; hence, BF and DF will coincide. Therefore, the angles B and D coincide, and are equal.

269. Corollary.—The three angles of an equilateral triangle are equal.

270. In an isosceles triangle, the angle included by the equal sides is usually called the *vertex* of the triangle, and the side opposite to it the *base*.

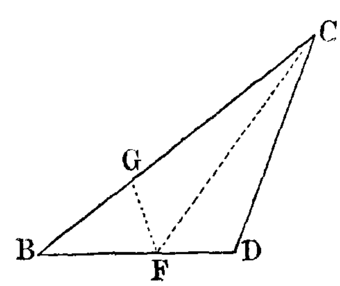
271. Corollary.—If a line pass through the vertex of an isosceles triangle, and also through the middle of the base, it will bisect the angle at the vertex, and be perpendicular to the base.

The straight line which has any two of these four conditions must have the other two (52).

UNEQUAL SIDES.

272. Theorem.—*When two sides of a triangle are unequal, the angle opposite to the greater side is greater than the angle opposite to the less side.*

If in the triangle BCD the side BC is greater than DC, then the angle D is greater than the angle B.



Let the line CF bisect the angle C, and be produced to the side BD. Then let the triangle CDF turn upon CF. CD will take the direction CB ; but, since CD is less than CB, the point D will fall between C and B, at G. Join GF.

Now, the angle FGC is equal to the angle D, because they coincide; and it is greater than the angle B, because it is exterior to the triangle BGF (261). Therefore, the angle D is greater than B.

273. Corollary.—When one side of a triangle is not the largest, the angle which is opposite to that side is acute (257).

274. Corollary.—In a scalene triangle, no two angles are equal.

EQUAL ANGLES.

275. Theorem.—*If two angles of a triangle are equal, the sides opposite them are equal.*

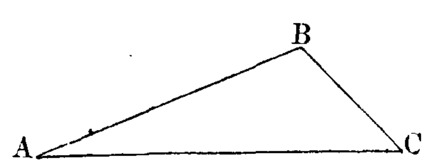
For if these sides were unequal, the angles opposite to them would be unequal (272), which is contrary to the hypothesis.

276. Corollary.—If a triangle is *equiangular*, that is, has all its angles equal, then it is equilateral.

UNEQUAL ANGLES.

277. Theorem.—*If two angles of a triangle are unequal, the side opposite to the greater angle is greater than the side opposite to the less.*

If, in the triangle ABC, the angle C is greater than the angle A, then AB is greater than BC.



For, if AB were not greater than BC, it would be either equal to it or less. If AB were equal to BC, the opposite angles A and C would be equal (268); and if AB were less than BC, then the angle C would be less than A (272); but both of these conclusions are contrary to the hypothesis. Therefore, AB being neither less than nor equal to BC, must be greater.

278. Corollary.—In an obtuse angled triangle, the longest side is opposite the obtuse angle; and in a right angled triangle, the longest side is opposite the right angle.

279. The HYPOTENUSE of a right angled triangle is the side opposite the right angle. The other two sides are called the *legs*.

The student will notice that some of the above propositions are but different statements of the principles of perpendicular and oblique lines.

EXERCISES.

280.—1. How many degrees are there in an angle of an equilateral triangle?

2. If one of the angles at the base of an isosceles triangle be double the angle at the vertex, how many degrees in each?

3. If the angle at the vertex of an isosceles triangle be double one of the angles at the base, what is the angle at the vertex?

4. To circumscribe a circle about a given triangle (149).

5. To inscribe a circle in a given triangle (252).

6. If two sides of a triangle be produced, the lines which bisect the two exterior angles and the third interior angle all meet in one point.

7. Draw a line DE parallel to the base BC of a triangle ABC, so that DE shall be equal to the sum of BD and CE.

8. Can a triangular field have one side 436 yards, the second 547 yards, and the third 984 yards long?

9. The angle at the base of an isosceles triangle being one-fourth of the angle at the vertex, if a perpendicular be erected to the base at its extreme point, and this perpendicular meet the opposite side of the triangle produced, then the part produced, the remaining side, and the perpendicular form an equilateral triangle.

10. If with the vertex of an isosceles triangle as a center, a circumference be drawn cutting the base or the base produced, then the parts intercepted between the curve and the extremities of the base, are equal.

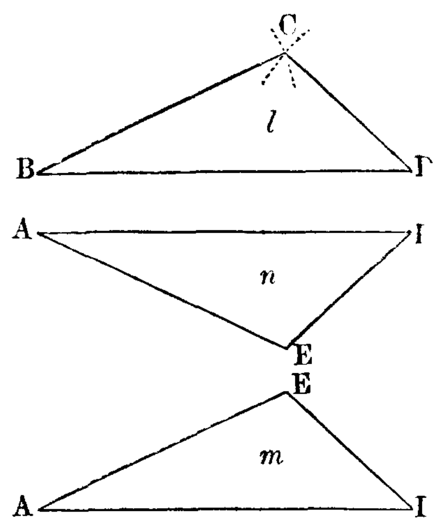
EQUALITY OF TRIANGLES.

281. The three sides and three angles of a triangle may be called its *six elements*. It may be shown that three of these are always necessary, and they are generally enough, to determine the triangle.

THREE SIDES EQUAL.

282. Theorem.—*Two triangles are equal when the three sides of the one are respectively equal to the three sides of the other.*

Let the side BD be equal to AI, the side BC equal to AE, and CD to EI; then the two triangles are equal.



Apply the line AI to its equal BD, so that the point A will fall upon B. Then I will fall upon D, since the lines are equal. Next, turn one of the triangles, if necessary, so that both shall fall on the same side of this common line.

Now, the point A being on B, the points E and C are at the same distance from B, and therefore they are both in the circumference, which has B for its center, and BC or AE for its radius (153). For a similar reason, the points E and C are both in the circumference, which has D for its center and DC or IE for its radius. These two circumferences have only one point common on one side of the line BD, which joins their centers (232). Hence, E and C are both at this point. Therefore (51), AE coincides with BC, and EI with CD; that is, the two triangles coincide throughout, and are equal.

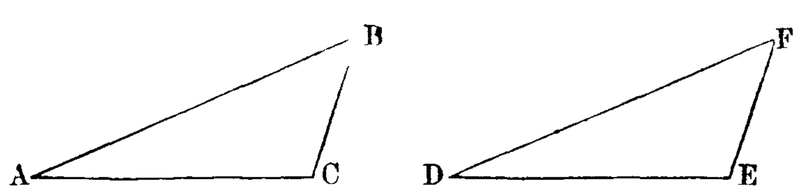
283. Every plane figure may be supposed to have two *faces*, which may be termed the upward and the downward faces. In order to place the triangle *m* upon *l*, we may conceive it to slide along the plane without turning over; but, in order to place *n* upon *l*, it must be turned over, so that its upward face will be upon the upward face of *l*.

There are, then, two methods of superposition; the first, called *direct*, when the downward face of one figure is applied to the upward face of the other; and the second, called *inverse*, when the upward faces of the two are applied to each other. Hitherto, we have used only the inverse method. Generally, in the chapter on the circumference, either method might be used indifferently.

TWO SIDES AND INCLUDED ANGLE.

284. Theorem.—*Two triangles are equal when they have two sides and the included angle of the one, respectively equal to two sides and the included angle of the other.*

If the angle A is equal to D, and the side AB to



the side DF, and AC to DE, then the two triangles are equal.

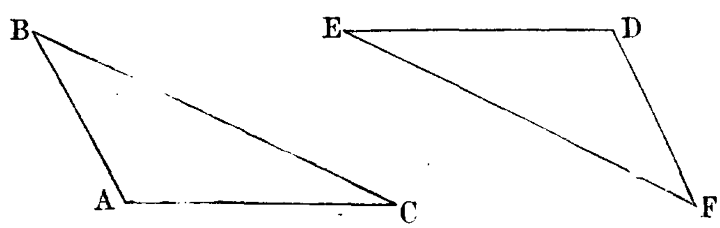
Apply the side AC to its equal DE, turning one triangle, if necessary, so that both shall fall upon the same side of that common line.

Then, since the angles A and D are equal, AB must take the direction DF, and these lines being equal, B will fall upon F. Therefore, BC and FE, having two points common, coincide; and the two triangles coincide throughout, and are equal.

ONE SIDE AND TWO ANGLES.

285. Theorem.—*Two triangles are equal when they have one side and two adjacent angles of the one, respectively equal to a side and the two adjacent angles of the other.*

If the triangles ABC and DEF have the side AC



equal to DE, and the angle A equal to D, and C equal to E, then the triangles are equal.

Apply the side AC to its equal DE, so that the vertices of the equal angles shall come together, A upon D, and C upon E, and turning one triangle, if necessary, so that both shall fall upon one side of the common line.

Then, since the angles A and D are equal, AB will take the direction DF, and the point B will fall somewhere in the line DF. Since the angles C and E are equal, CB will take the direction EF, and B will also be in the line EF. Therefore, B falls upon F, the only point common to the two lines DF and EF. Hence, the sides of the one triangle coincide with those of the other, and the two triangles are equal.

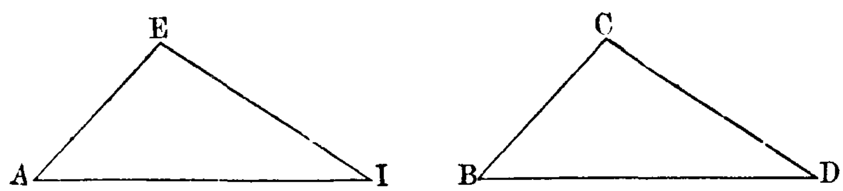
286. Theorem.—*Two triangles are equal when they have one side and any two angles of the one, respectively equal to the corresponding parts of the other.*

For the third angle of the first triangle must be equal to the third angle of the other (262). Then this becomes a case of the preceding theorem.

TWO SIDES AND AN OPPOSITE ANGLE.

287. Theorem.—*Two triangles are equal when one of them has two sides, and the angle opposite to the side which is equal to or greater than the other, respectively equal to the corresponding parts of the other triangle.*

Let the sides AE and EI, EI being equal to or



greater than AE, and the angle A, be respectively equal to the sides BC, CD, and the angle B. Then the triangles are equal.

For the side AE may be placed on its equal BC. Then, since the angles A and B are equal, AI will take the direction BD, and the points I and D will both be in the common line BD. Since EI and CD are equal, the points I and D are both in the circumference whose center is at C, and whose radius is equal to CD. Now, this circumference cuts a straight line extending from B toward D in only one point; for B is either within or on the circumference, since BC is equal to or less than CD. Therefore, I and D are both at that point.

Hence, AI and BD are equal, and the triangles are equal (282).

288. Corollary.—Two triangles are equal when they have an obtuse or a right angle in the one, together with the side opposite to it, and one other side, respectively equal to those parts in the other triangle (278).

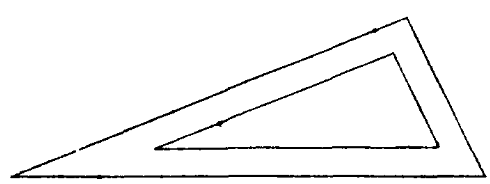
The two following are corollaries of the last five theorems, and of the definition (40).

289. Corollary.—In equal triangles each part of one is equal to the corresponding part of the other.

290. Corollary.—In equal triangles the equal parts are similarly arranged, so that equal angles are opposite to equal sides.

EXCEPTIONS TO THE GENERAL RULE.

291. A general rule as to the equality of triangles has been given (281). There are two exceptions.

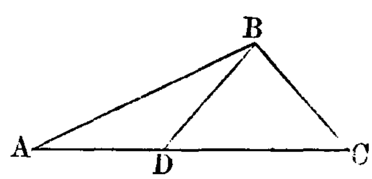


1. When the three angles are given.

For two very unequal triangles may have the angles of one equal to those of the other.

2. When two unequal sides and the angle opposite to the less are given.

For with the sides AB and BC and the angle A given, there are two triangles, ABC and ABD.

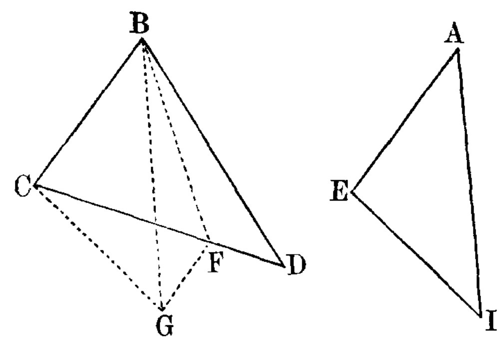


292. The student may show that two parts alone are never enough to determine a triangle.

UNEQUAL TRIANGLES.

293. Theorem.—*When two triangles have two sides of the one respectively equal to two sides of the other, and the included angles unequal, the third side in that triangle which has the greater angle, is greater than in the other.*

Let BCD and AEI be two triangles, having BC equal to AE, and BD equal to AI, and the angle A less than B. Then, it is to be proved that CD is greater than EI.



Apply the triangle AEI to BCD, so that AE will coincide with its equal BC. Since the angle A is less than B, the side AI will fall within the angle CBD. Let BG be its position, and EI will fall upon CG. Then let a line BF bisect the angle GBD. Join FG.

The triangles GBF and BDF have the side BF common, the side GB equal to the side DB, since each is equal to AI, and the included angles GBF and DBF equal by construction. Therefore, the triangles are equal (284), and FG is equal to FD (289). Hence, CD, the sum of CF and FD, is equal to the sum of CF and FG (7), which is greater than CG (54). Therefore, CD is greater than CG, or its equal EI.

If the point I should fall within the triangle BCD or on the line CD, the demonstration would not be changed.

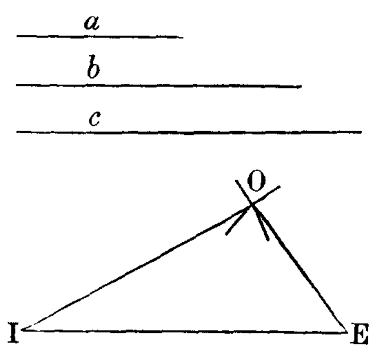
294. Theorem.—*Conversely, if two triangles have two sides of the one equal to two sides of the other, and the third sides unequal, then the angles opposite the third sides are unequal, and that is greater which is opposite the greater side.*

For if it were less, then the opposite side would be less (293), and if it were equal, then the opposite sides would be equal (284); both of which are contrary to the hypothesis.

PROBLEMS IN DRAWING.

295. Problem.—*To draw a triangle when the three sides are given.*

Let a, b, and c be the given lines. Draw the line IE equal to c. With I as a center, and with the line b as a radius, describe an arc, and with E as a center and the line a as a radius, describe a second arc, so that the two may cut each other. Join O, the point of intersection of these arcs, with I and with E. IOE is the required triangle.



If c were greater than the sum of a and b, what would have been the result?

What, if c were less than the difference of a and b?

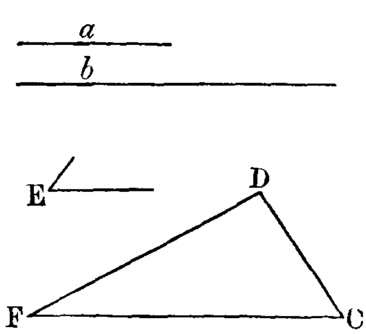
Has the problem more than one solution; that is, can unequal triangles be drawn which comply with the conditions? Why?

296. Corollary.—In the same way, draw a triangle equal to a given triangle.

297. Problem.—*To draw a triangle, two sides and the included angle being given.*

Let a and b be the given lines, and E the angle.

Draw FC equal to b. At C make an angle equal to E. Take DC equal to a, and join FD. Then FDC is a triangle having the required conditions.



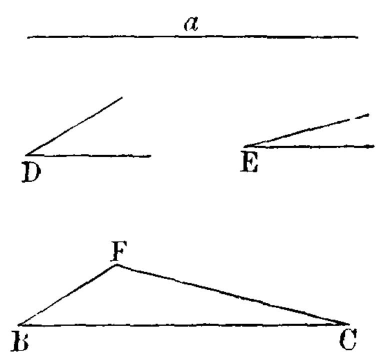
Has this problem more than one solution? Why?

Is this problem always soluble, whatever may be the size of the given angle, or the length of the given lines? Why?

298. Problem.—*To draw a triangle when one side and the adjacent angles are given.*

Let a be the given line, and D and E the angles.

Draw BC equal to a. At B make an angle equal to D, and at C an angle equal to E. Produce the sides till they meet at the point F. FBC is a triangle having the given side and angles.



Has this problem more than one solution?

Can it be solved, whatever be the given angles, or the given line?

299. Problem.—*To draw a triangle when one side and two angles are given.*

If one of the angles is opposite the given side, find the supplement of the sum of the given angles (214). This will be the other adjacent angle (256). Then proceed as in Article 298.

300. Problem.—*To draw a triangle when two sides and an angle opposite to one of them are given.*

Construct an angle equal to the given angle. Lay off on one side of the angle the length of the given adjacent side. With the extremity of this adjacent side as a center, and with a radius equal to the side opposite the given angle, draw an arc. This arc may cut the opposite side of the angle. Join the point of intersection with the end of the adjacent side which was taken as a center. A triangle thus formed has the required conditions.

The student can better discuss this problem after drawing several triangles with various given parts. Let the given angle vary from very obtuse to very acute; and let the opposite side vary from being much larger to much smaller than the side adjacent to the given angle. Then let the student explain when this problem has only one solution, when it has two, and when it can not be solved.

EXERCISES.

301.—1. The base of an isosceles triangle is to one of the other sides as three to two. Find by construction and measurement, whether the vertical angle is acute or obtuse.

2. Two right angled triangles are equal, when any two sides of the one are equal to the corresponding sides of the other.

3. Two right angled triangles are equal, when an acute angle and any side of the one are equal to the corresponding parts of the other.

4. Divide a given triangle into four equal parts.

5. Construct a right angled triangle when,

    I. An acute angle and the adjacent leg are given;
    II. An acute angle and the opposite leg are given;
    III. A leg and the hypotenuse are given;
    IV. When the two legs are given.

SIMILAR TRIANGLES.

302. Similar magnitudes have been defined to be those which have the same form while they differ in extent (37).

303. Let the student bear in mind that the form of a figure depends upon the relative directions of its points, and that angles are differences in direction. Therefore, the definition may be stated thus:

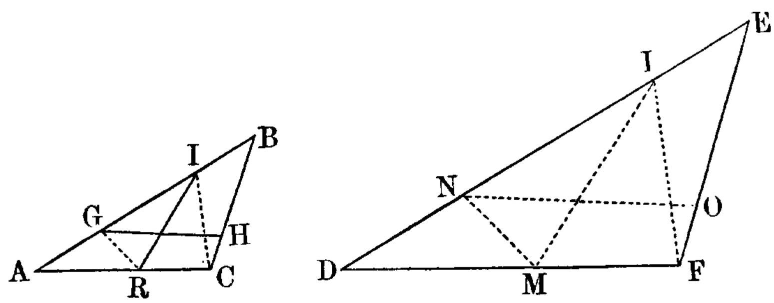
Two figures are similar when every angle that can be formed by lines joining points of one, has its corresponding equal and similarly situated angle in the other.

ANGLES EQUAL.

304. Theorem.—*Two triangles are similar, when the three angles of the one are respectively equal to the three angles of the other.*

This may appear to be only a case of the definition of similar figures; but it may be shown that every angle that can be made by any lines whatever in the one, may have its corresponding equal and similarly situated angle in the other.

Let the angles A, B, and C be respectively equal to



the angles D, E, and F. Suppose GH and IR to be any two lines in the triangle ABC.

Join IC and GR. From F, the point homologous to C, extend FL, making the angle LFE equal to ICB.

Now, the triangles LFE and ICB have the angles B and E equal, by hypothesis, and the angles at C and F equal, by construction. Therefore, the third angles, ELF and BIC, are equal (262). By subtraction, the angles AIC and DLF are equal, and the angles ACI and DFL.

From L extend LM, making the angle FLM equal to CIR. Then the two triangles FLM and CIR have the angles at C and F equal, as just proved, and the angles at I and L equal, by construction. Therefore, the third angles, LMF and IRC, are equal.

Join RG. Construct MN homologous to RG, and NO homologous to GH. Then show, by reasoning in the same manner, that the angles at N are equal to the corresponding angles at G; and so on, throughout the two figures.

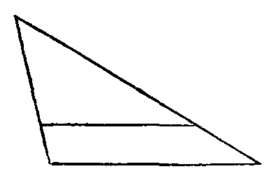
The demonstration is similar, whatever lines be first made in one of the triangles.

Therefore, the relative directions of all their points are the same in both triangles; that is, they have the same form. Therefore, they are similar figures.

305. Corollary.—Two similar triangles may be divided into the same number of triangles respectively similar, and similarly arranged.

306. Corollary.—Two triangles are similar, when two angles of the one are respectively equal to two angles of the other. For the third angles must be equal also (262).

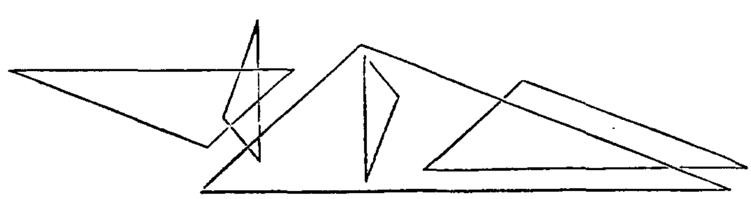
307. Corollary.—If two sides of a triangle be cut by a line parallel to the third side, the triangle cut off is similar to the original triangle (124).



308. Theorem.—*Two triangles are similar, when the sides of one are parallel to those of the other; or, when the sides of one are perpendicular to those of the other.*

We know (138 and 139) that the angles formed by lines which are parallel are either equal or supplementary; and that the same is true of angles whose sides are perpendicular (140). We will show that the angles can not be supplementary in two triangles.

If even two angles of one triangle could be respect-

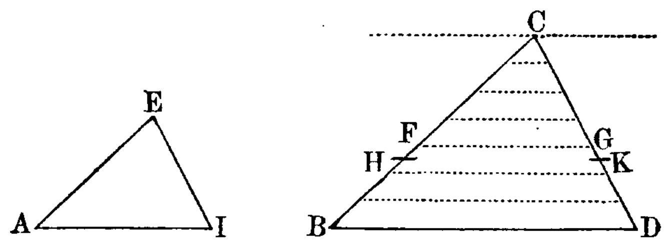


ively supplementary to two angles of another, the sum of these four angles would be four right angles; and then the sum of all the angles of the two triangles would be more than four right angles, which is impossible (255). Hence, when two triangles have their sides respectively parallel or perpendicular, at least two of the angles of one triangle must be equal to two of the other. Therefore, the triangles are similar (306).

SIDES PROPORTIONAL.

309. Theorem.—*One side of a triangle is to the homologous side of a similar triangle as any side of the first is to the homologous side of the second.*

If AE and BC are homologous sides of similar tri-



angles, also EI and CD, then,

AE : BC : : EI : CD.

Take CF equal to EA, and CG equal to EI, and join FG. Then the triangles AEI and FCG are equal (284), and the angles CFG and CGF are respectively equal to the angles A and I, and therefore equal to the angles B and D. Hence, FG is parallel to BD (129). Let a line extend through C parallel to FG and BD.

Suppose BC divided at the point F into parts which have the ratio of two whole numbers, for example, four and three. Then let the line CF be divided into four, and BF into three equal parts. Let lines parallel to BD extend from the several points of division till they meet CD.

Since BC is divided into equal parts, the distances between these parallels are all equal (135). Therefore, CD is also divided into seven equal parts (134), of which CG has four. That is,

CF : CB : : CG : CD : : 4 : 7.

But if the lines BC and CF have not the ratio of two whole numbers, then let BC be divided into any number of equal parts, and a line parallel to BD pass through H, the point of division nearest to F. Such a line must divide CD and CB proportionally, as just proved; that is,

CH : CB : : CK : CD.

By increasing the number of the equal parts into which BC is divided, the points H and K may be made to approach within any conceivable distance of F and G. Therefore, CF and CG are the limits of those lines, CH and CK, which are commensurable with BC and CD; and we may substitute CF and CG in the last proportion for CH and CK.

Hence, whatever be the ratio of CF to CB, it is the same as that of CG to CD. By substituting for CF and CG the equal lines AE and EI, we have,

AE : BC : : EI : CD.

By similar reasoning it may be shown that

AI : BD : : EI : CD.

310. Corollary.—The ratio is the same between any two homologous lines of two similar triangles.

311. This ratio of any side of a triangle to the homologous side of a similar triangle, is called the *linear ratio* of the two figures.

312. Corollary.—The perimeters of similar triangles have the linear ratio of the two figures. For,

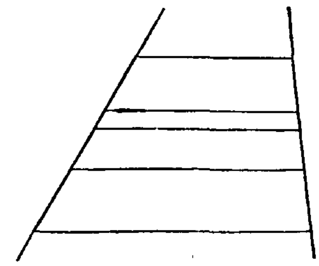
AE : BC : : EI : CD : : IA : DB.

Therefore (23),

AE+EI+IA : BC+CD+DB : : AE : BC.

313. Corollary.—If two sides of a triangle are cut by one or more lines parallel to the third side, the two sides are cut proportionally. For the triangles so formed are similar (307).

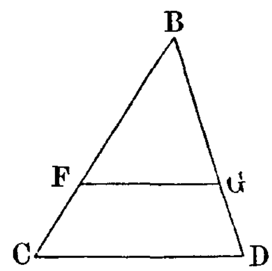
314. Corollary.—When several parallel lines are cut by two secants, the secants are divided proportionally.



For the secants being produced till they meet, form several similar triangles.

315. Theorem.—*If two sides of a triangle be cut proportionally by a straight line, the secant line is parallel to the third side.*

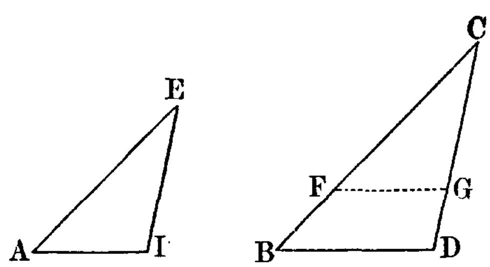
Let BCD be the triangle, and FG the secant.



A line parallel to CD may pass through F, and such a line must divide BD in the same ratio as BC (313). But, by hypothesis, BD is so divided at the point G. Therefore, a line through F parallel to CD, must pass through G, and coincide with FG. Hence, FG is parallel to CD.

316. Theorem.—*Two triangles are similar when the ratios between each side of the one and a corresponding side of the other are the same.*

Suppose AE : BC : : EI : CD : : AI : BD.



Take CF equal to EA and CG equal to EI, and join FG. Then,

CF : CB : : CG : CD.

Therefore, FG is parallel to BD (315), the triangles CFG and CBD are similar (307), and

CF : CB : : FG : BD.

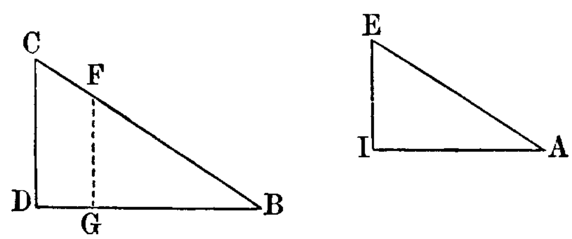
But, by hypothesis,

EA : CB :: AI : BD.

Hence, since CF is equal to EA, FG is equal to AI, and the triangles AEI and FCG are equal. Therefore, the triangles AEI and BCD have their angles equal, and are similar.

317. Theorem.—*Two triangles are similar when two sides of the one have respectively to two sides of the other the same ratio, and the included angles are equal.*

Suppose AE : BC :: AI : BD; and let the angle A be equal to B.



Take BF equal to AE, and BG equal to AI, and join FG. Then the triangles AEI and BFG are equal (284), and the angle BFG is equal to E, and BGF is equal to I. Since the sides of the triangle BCD are cut proportionally by FG, the angle BFG is equal to C, and BGF is equal to D (315). Therefore, the triangles AEI and BCD are mutually equiangular and similar.

318. If two similar triangles have two homologous lines equal, since all other homologous lines have the same ratio, they must also be equal, and consequently the two figures are equal. Thus, the equality of figures may be considered as a case of similarity.

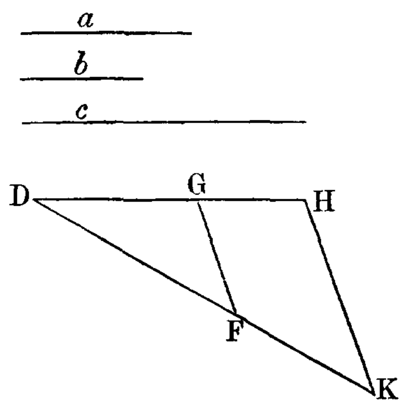
PROBLEMS IN DRAWING.

319. Problem.—*To find a fourth proportional to three given straight lines.*

Let a be the given extreme, and b and c the given means.

Take DG equal to a, the given extreme. Produce it, making

DH equal to c, one of the means. From G draw GF equal to b. Then, from D draw a line through F, and from H a line parallel to GF. Produce these two lines till they meet at the point K. HK is the required fourth proportional.



For the triangles DGF and DHK are similar (307). Hence,

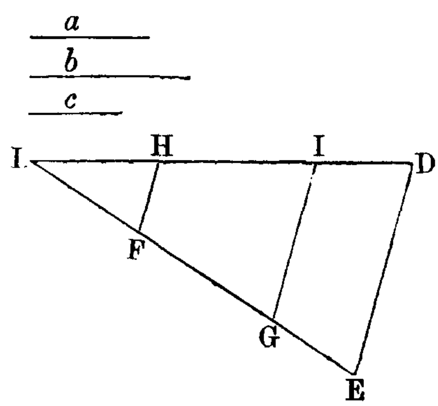
DG : GF : : DH : HK.

That is, $a : b :: c :$ HK.

It is most convenient to make GF and HK perpendicular to DH.

320. Problem.—*To divide a given line into parts having a certain ratio.*

Let LD be the line to be divided into parts proportional to the lines a, b, and c.



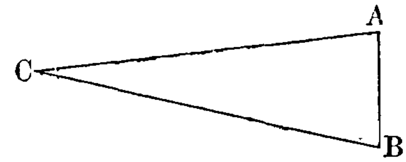
From L draw the line LE equal to the sum of a, b, and c, making LF equal to a, FG equal to b, and GE equal to c. Join DE, and draw GI and FH parallel to DE. LH, HI, and ID are the parts required.

The demonstration is similar to the last.

321. Problem.—*To divide a given line into any number of equal parts.*

This may be done by the last problem; but when the given line is small, the following method is preferable.

To divide the line AB into ten equal parts; draw AC indefinitely, and take on it ten equal parts. Join BC, and from the several points of division of AC, draw lines parallel to AB, and produce them to BC. The parallel nearest to AB is nine-tenths of AB, the next is eight-tenths, and so on.



This also depends upon similarity of triangles.

322. Problem.—*To draw a triangle on a given base, similar to a given triangle.*

Let this problem be solved by the student.

RIGHT ANGLED TRIANGLES.

323. Every triangle may be divided into two right angled triangles, by a perpendicular let fall from one of its vertices upon the opposite side. Thus the investigation of the properties of right angled triangles leads to many of the properties of triangles in general.

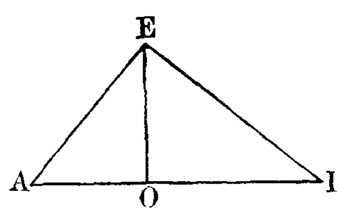
324. Theorem.—*If in a right angled triangle, a perpendicular be let fall from the vertex of the right angle upon the hypotenuse, then,*

1. *Each of the triangles thus formed is similar to the original triangle;*

2. *Either leg of the original triangle is a mean proportional between the hypotenuse and the adjacent segment of the hypotenuse; and,*

3. *The perpendicular is a mean proportional between the two segments of the hypotenuse.*

The triangles AEO and AEI have the angle A common, and the angles AEI and AOE are equal, being right angles. Therefore, these two triangles are similar (306)



That the triangles EOI and EIA are similar, is proved by the same reasoning.

Since the triangles are similar, the homologous sides are proportional, and we have

$$AI : AE :: AE : AO;$$

That is, the leg AE is a mean proportional between the whole hypotenuse and the segment AO which is adjacent to that leg.

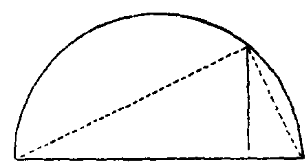
In like manner, prove that EI is a mean proportional between AI and OI.

Lastly, the triangles AEO and EIO are similar (304), and therefore,

$$AO : OE :: OE : OI.$$

That is, the perpendicular is a mean proportional between the two segments of the hypotenuse.

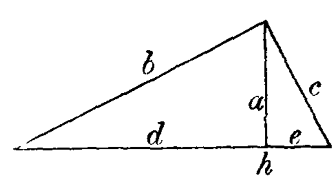
325. Corollary.—A perpendicular let fall from any point of a circumference upon a diameter, is a mean proportional between the two segments which it makes of the diameter.



326. In the several proportions just demonstrated, in place of the lines we may substitute those numbers which constitute the ratios (14). Indeed, it is only upon this supposition that the proportions have a meaning. It is the same whether these numbers be integers or radicals, since we know that the terms of the ratio are in fact numbers.

327. Theorem.—*The second power of the length of the hypotenuse is equal to the sum of the second powers of the lengths of the two legs of a right angled triangle.*

Let h be the hypotenuse, a the perpendicular let fall upon it, b and c the legs, and d and e the corresponding segments of the hypotenuse made by the perpendicular. That is, these letters represent the number of times, whether integral or not, which some unit of length is contained in each of these lines.



By the second conclusion of the last theorem, we have

$$h : b :: b : d, \quad \text{and} \quad h : c :: c : e.$$

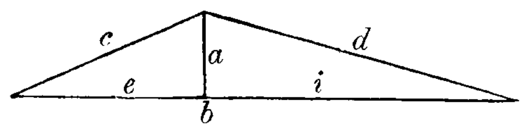
Hence, (16), $hd = b^2$, and $he = c^2$.

By adding these two, $h(d + e) = b^2 + c^2$.

But $d + e = h$. Therefore, $h^2 = b^2 + c^2$.

328. Theorem.—*If, in any triangle, a perpendicular be let fall from one of the vertices upon the opposite side as a base, then the whole base is to the sum of the other two sides, as the difference of those sides is to the difference of the segments of the base.*

Let a be the perpendicular, b the base, c and d the sides, and e and i the segments of the base.



Then, two right angled triangles are formed, in one of which we have

$$a^2 + i^2 = d^2;$$

and in the other, $a^2 + e^2 = c^2$.

Subtracting, $i^2 - e^2 = d^2 - c^2$.

Factoring, $(i + e)(i - e) = (d + c)(d - c)$.

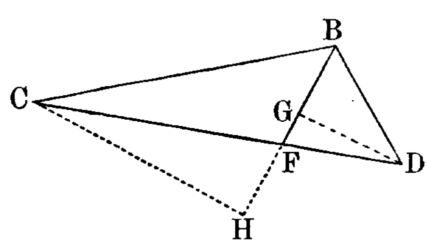
Whence (18), $i + e : d + c :: d - c : i - e$.

329. Theorem.—*If a line bisects an angle of a triangle, it divides the opposite side in the ratio of the adjacent sides.*

If BF bisects the angle CBD, then

$$CF : FD :: CB : BD.$$

This need be demonstrated only in the case where the sides adjacent to the bisected angle are not equal.



From C and from D, let perpendiculars DG and CH fall upon BF, and BF produced.

Then, the triangles BDG and BCH are similar, for they have equal angles at B, by hypothesis, and at G and H, by construction. Hence,

CB : BD : : CH : DG.

But the triangles DGF and CHF are also mutually equiangular and similar. Hence,

CF : FD : : CH : DG.

Therefore (21), CF · FD : : CB : BD.

330. Problem in Drawing.—*To find a mean proportional to two given straight lines.*

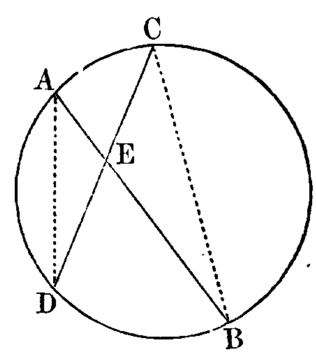
Make a straight line equal to the sum of the two. Upon this as a diameter, describe a semi-circumference. Upon this diameter, erect a perpendicular at the point of meeting of the two given lines. Produce this to the circumference. The line last drawn is the required line.

Let the student construct the figure and demonstrate.

CHORDS, SECANTS, AND TANGENTS.

331. Theorem.—*If two chords of a circle cut each other, the parts of one may be the extremes, and the parts of the other the means, of a proportion.*

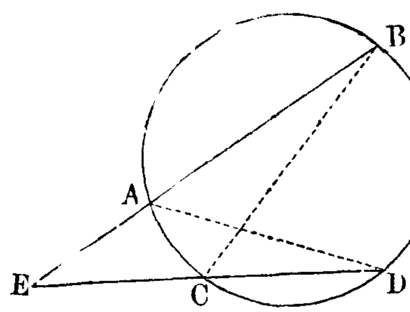
Join AD and CB. Then the two triangles AED and CEB have the angle A equal to the angle C, since they are inscribed in the same arc (224). For the same reason, the angles D and B are equal. Therefore, the triangles are similar (306); and we have (309),



AE : EC : : DE : EB.

332. Theorem.—*If from the same point, without a circle, two lines cutting the circumference extend to the farther side, then the whole of one secant and its exterior* *part may be the extremes, and the whole of the other secant and its exterior part may be the means, of a proportion.*

Joining BC and AD, the triangles AED and CEB are similar; for they have the angle E common, and the angles at B and D equal. Therefore,



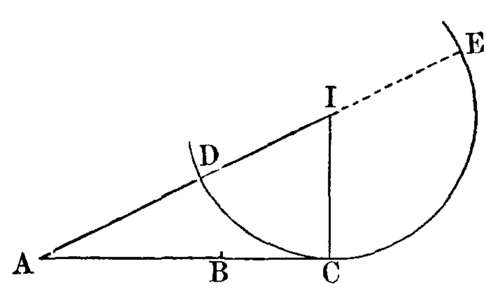
AE : EC : : DE : EB.

333. Corollary.—If from the same point there be a tangent and a secant, the tangent is a mean proportional between the secant and its exterior part. For the tangent is the limit of all the secants which pass through the point of meeting.

334. Problem in Drawing.—*To divide a given straight line into two parts so that one of them is a mean proportional between the whole line and the other part.*

This is called dividing a line in *extreme and mean ratio.*

Let AC be the given line. At C erect a perpendicular, CI, equal to half of AC. Join AI. Take ID equal to CI, and then AB equal to AD. The line AC is divided at the point B in extreme and mean ratio. That is,



AC : AB : : AB : BC.

With I as a center and IC as a radius, describe an arc DCE, and produce AI till it meets this arc at E. Then, AC is a tangent to this arc (178), and therefore (333),

AE : AC : : AC : AD.

Or (24), AE—AC : AC : : AC—AD : AD.

But AC is twice IC, by construction, and DE is twice IC, because DE is a diameter and IC is a radius. Therefore, the first term of the last proportion, AE — AC, is equal to AE — DE, which is AD; but AD is, by construction, equal to AB. Also, the third term, AC — AD, is equal to AC — AB, which is BC. And the fourth term is equal to AB. Substituting these equals, the proportion becomes

AB : AC :: BC : AB.

By inversion (19), AC : AB :: AB : BC.

ANALYSIS AND SYNTHESIS.

335. Geometrical Analysis is a process employed both for the discovery of the solution of problems and for the investigation of the truth of theorems. Analysis is the reverse of synthesis. Synthesis commences with certain principles, and proceeds by undeniable and successive inferences. The whole theory of geometry is an example of this method.

In the *analysis* of a problem, what was required to be done is supposed to have been effected, and the consequences are traced by a series of geometrical constructions and reasonings, till at length they terminate in the data of the problem, or in some admitted truth. See suggestions, Article 245.

In the *synthesis* of a problem, however, the last consequence of the analysis is the first step of the process, and the solution is effected by proceeding in a contrary order through the several steps of the analysis, until the process terminates in the thing required to be done.

If, in the analysis, we arrive at a consequence which conflicts with any established principle, or which is inconsistent with the data of the problem, then the solution is impossible. If, in certain relations of the given magnitudes, the construction is possible, while in other relations it is impossible, the discovery of these relations is a necessary part of the discussion of the problem.

In the analysis of a theorem, the question to be determined is, whether the proposition is true, as stated; and, if so, how this truth is to be demonstrated. To do this, the truth is assumed, and the successive consequences of this assumption are deduced till they terminate in the hypothesis of the theorem, or in some established truth.

The theorem will be proved synthetically by retracing, in order, the steps of the investigation pursued in the analysis, till they terminate in the conclusion which had been before assumed. This constitutes the demonstration.

If, in the analysis, the assumption of the truth of the proposition leads to some consequence which conflicts with an established principle, the false conclusion thus arrived at indicates the falsehood of the proposition which was assumed to be true.

In a word, analysis is used in geometry in order to discover truths, and synthesis to demonstrate the truths discovered.

Most of the problems and theorems which have been given for EXERCISES, are of so simple a character as scarcely to admit of the principle of geometrical analysis being applied to their solution.

336. A problem is said to be *determinate* when it admits of one definite solution; but when the same construction may be made on the other side of any given line, it is not considered a different solution. A problem is *indeterminate* when it admits of more than one definite solution. Thus, Article 300 presents a case where the problem may be determinate, indeterminate, or insolvable, according to the size of the given angle and extent of the given lines.

The solution of an indeterminate problem frequently amounts to finding a geometrical locus; as, to find a point equidistant from two given points; or, to find a point at a given distance from a given line.

EXERCISES.

337. Nearly all the following exercises depend upon principles found in this chapter, but a few of them depend on those of previous chapters.

1. If there be an isosceles and an equilateral triangle on the same base, and if the vertex of the inner triangle is equally distant from the vertex of the outer one and from the ends of the base, then, according as the isosceles triangle is the inner or the outer one, its base angle will be $\frac{1}{4}$ of, or $2\frac{1}{2}$ times the vertical angle.

2. The semi-perimeter of a triangle is greater than any one of the sides, and less than the sum of any two.

3. Through a given point, draw a line such that the parts of it, between the given point and perpendiculars let fall on it from two other given points, shall be equal.

What would be the result, if the first point were in the straight line joining the other two?

4. Of all triangles on the same base, and having their vertices in the same line parallel to the base, the isosceles has the greatest vertical angle.

5. If, from a point without a circle, two tangents be made to the circle, and if a third tangent be made at any point of the circumference between the first two, then, at whatever point the last tangent be made, the perimeter of the triangle formed by these tangents is a constant quantity.

6. Through a given point between two given lines, to draw a line such that the part intercepted by the given lines shall be bisected at the given point.

7. From a point without two given lines, to draw a line such that the part intercepted between the given lines shall be equal to the part between the given point and the nearest line.

8. The middle point of a hypotenuse is equally distant from the three vertices of a right angled triangle.

9. Given one angle, a side adjacent to it, and the difference of the other two sides, to construct the triangle.

10. Given one angle, a side opposite to it, and the difference of the other two sides, to construct the triangle.

11. Given one angle, a side opposite to it, and the sum of the other two sides, to construct the triangle.

12. Trisect a right angle.

13. If a circle be inscribed in a right angled triangle, the difference between the hypotenuse and the sum of the two legs is equal to the diameter of the circle.

14. If from a point within an equilateral triangle, a perpendicular line fall on each side, the sum of these perpendiculars is a constant quantity.

How should this theorem be stated, if the point were outside of the triangle?

15. Find the locus of the points such that the sum of the distances of each from the two sides of a given angle, is equal to a given line.

16. Find the locus of the points such that the difference of the distances of each from two sides of a given angle, is equal to a given line.

17. Demonstrate the last corollary (333) by means of similar triangles.

18. To draw a tangent common to two given circles.

19. To construct an isosceles triangle, when one side and one angle are given.

20. If in a right angled triangle one of the acute angles is equal to twice the other, then the hypotenuse is equal to twice the shorter leg.

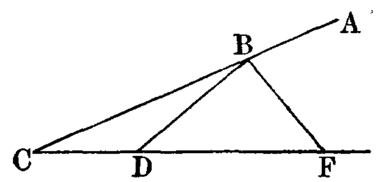
21. Draw a line DE parallel to the base BC of a triangle ABC, so that DE shall be equal to the difference of BD and CE.

22. In a given circle, to inscribe a triangle similar to a given triangle.

23. In a given circle, find the locus of the middle points of those chords which pass through a given point.

24. To describe a circumference tangent to three given equal circumferences, which are tangent to each other.

25. If a line bisects an exterior angle of a triangle, it divides the base produced into segments which are proportional to the adjacent sides. That is, if BF bisects the angle ABD, then,



CF : FD :: CB : BD.

26. The parts of two parallel lines intercepted by several straight lines which meet at one point, are proportional.

The converging lines are also divided in the same ratio.

27. Two triangles are similar, when two sides of one are proportional to two sides of the other, and the angle opposite to that side which is equal to or greater than the other given side in one, is equal to the homologous angle in the other.

28. The perpendiculars erected upon the several sides of a triangle at their centers, meet in one point.

29. The lines which bisect the several angles of a triangle, meet in one point.

30. The altitudes of a triangle, that is, the perpendiculars let fall from the several vertices on the opposite sides, meet in one point.

31. The lines which join the several vertices of a triangle with the centers of the opposite sides, meet in one point.

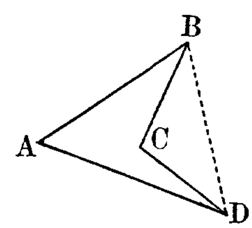
32. Each of the lines last mentioned is divided at the point of meeting into two parts, one of which is twice as long as the other.

CHAPTER VI.

QUADRILATERALS.

338. In a polygon, two angles which immediately succeed each other in going round the figure, are called *adjacent* angles. The student will distinguish adjacent angles of a polygon from the adjacent angles defined in Article 85.

A Diagonal of a polygon is a straight line joining the vertices of any two angles which are not adjacent. Sometimes a diagonal is exterior, as the diagonal BD of the figure ABCD.



A Convex polygon has all its diagonals interior.

A Concave polygon has at least one diagonal exterior, as in the above diagram.

Angles, such as BCD, are called *reëntrant.*

339. A Quadrilateral is a polygon of four sides.

340. Corollary.—Every quadrilateral has two diagonals.

341. Corollary.—An interior diagonal of a quadrilateral divides the figure into two triangles.

EQUAL QUADRILATERALS.

342. Theorem.—*Two quadrilaterals are equal when they are each composed of two triangles, which are respectively equal, and similarly arranged.*

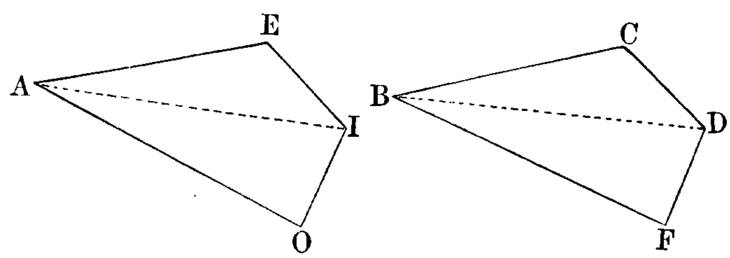
For, since the parts are equal and similarly arranged, the wholes may be made to coincide (40).

343. Corollary.—Conversely, two equal quadrilaterals may be divided into equal triangles similarly arranged. In every convex quadrilateral this division may be made in either of two ways.

344. Theorem.—*Two quadrilaterals are equal when the four sides and a diagonal of one are respectively equal to the four sides and the same diagonal of the other.*

By the same diagonal is meant the diagonal that has the same position with reference to the equal sides.

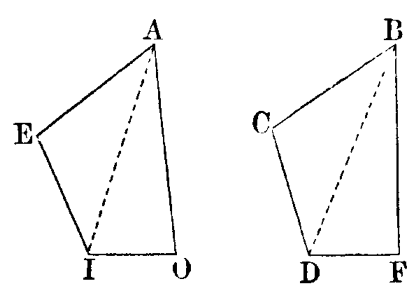
For, since all their sides are equal, the triangles AEI



and BCD are equal, also the triangles AIO and BDF (282). Therefore, the quadrilaterals are equal (342).

345. Theorem. — *Two quadrilaterals are equal when the four sides and an angle of the one are respectively equal to the four sides and the similarly situated angle of the other.*

By the similarly situated angle is meant the angle included by equal sides.



For, if the sides AE, IE, and the included angle E are respectively equal to the side BC, DC, and the included angle C, then the triangles AEI and BCD are equal (284); and AI is equal to BD. But since the three sides of the triangles AIO and BDF are respectively equal, the triangles are equal (282). Hence, the quadrilaterals are equal (342).

SUM OF THE ANGLES.

346. Theorem.—*The sum of the angles of a quadrilateral is equal to four right angles.*

For the angles of the two triangles into which every quadrilateral may be divided, are together coincident with the angles of the quadrilateral. Therefore, the sum of the angles of a quadrilateral is twice the sum of the angles of a triangle.

Let the student illustrate this with a diagram.

In applying this theorem to a concave figure (338), the value of the reëntrant angle must be taken on the side toward the polygon, and therefore as amounting to more than two right angles.

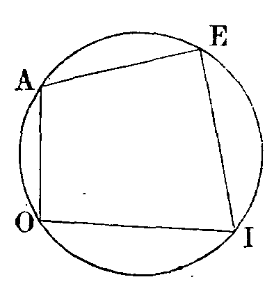
INSCRIBED QUADRILATERAL.

347. Problem.—*Any four points of a circumference may be joined by chords, thus making an inscribed quadrilateral.*

This is a corollary of Article 47.

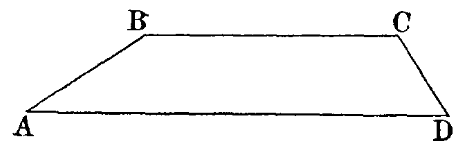
348. Theorem.—*The opposite angles of an inscribed quadrilateral are supplementary.*

For the angle A is measured by half of the arc EIO (222), and the angle I by half of the arc EAO. Therefore, the two together are measured by half of the whole circumference, and their sum is equal to two right angles (207).



TRAPEZOID.

349. If two adjacent angles of a quadrilateral are supplemental, the remaining angles are also supplemental (346). Then, one pair of opposite sides must be parallel (131).

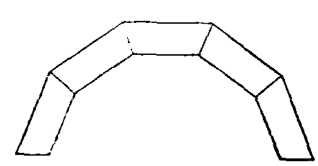


A TRAPEZOID is a quadrilateral which has two sides parallel. The parallel sides are called its *bases*.

350. Corollary.—If the angles adjacent to one base of a trapezoid be equal, those adjacent to the other base must also be equal. For if A and D are equal, their supplements, B and C, must be equal (96).

APPLICATION.

351. The figure described in the last corollary is symmetrical. For it can be divided into equal parts by a line joining the middle points of the bases.



The symmetrical trapezoid is used in architecture, sometimes for ornament, and sometimes as the form of the stones of an arch.

EXERCISES.

352.—1. To construct a quadrilateral when the four sides and one diagonal are given. For example, the side AB, 2 inches; the side BC, 5; CD, 3; DA, 4; and the diagonal AC, 6 inches.

2. To construct a quadrilateral when the four sides and one angle are given.

3. In a quadrilateral, join any point on one side to each end of the side opposite, and with the figure thus constructed demonstrate the theorem, Article 346.

4. The sum of two opposite sides of any quadrilateral which is

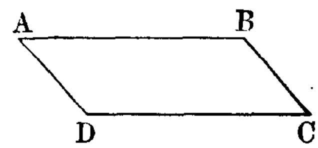
circumscribed about a circle, is equal to the sum of the other two sides.

5. If the two oblique sides of a trapezoid be produced till they meet, then the point of meeting, the point of intersection of the two diagonals of the trapezoid, and the middle points of the two bases are all in one straight line.

PARALLELOGRAMS.

353. A Parallelogram is a quadrilateral which has its opposite sides parallel.

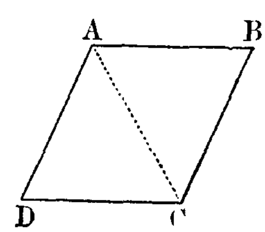
354. Corollary.—Two adjacent angles of a parallelogram are supplementary. The angles A and B, being between the parallels AD and BC, and on one side of the secant AB, are supplementary (126).



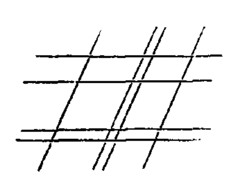
355. Corollary.—The opposite angles of a parallelogram are equal. For both D and B are supplements of the angle C (96).

356. Theorem.—*The opposite sides of a parallelogram are equal.*

For, joining AC by a diagonal, the triangles thus formed have the side AC common; the angles ACB and DAC equal, for they are alternate (125); and ACD and BAC equal, for the same reason. Therefore (285), the triangles are equal, and the side AD is equal to BC, and AB to CD.



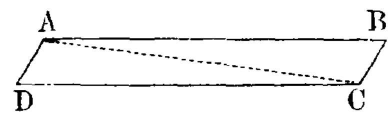
357. Corollary.—When two systems of parallels cross each other, the parts of one system included between two lines of the other are equal.



358. Corollary.—A diagonal divides a parallelogram into two equal triangles. But the diagonal does not divide the figure symmetrically, because the position of the sides of the triangles is reversed.

359. Theorem.—*If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

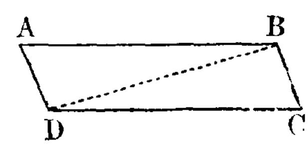
Join AC. Then, the triangles ABC and CDA are equal. For the side AD is equal to BC, and DC is equal to AB, by hypothesis; and they have the side AC common. Therefore, the angles DAC and BCA are equal. But these angles are alternate with reference to the lines AD and BC, and the secant AC. Hence, AD and BC are parallel (130), and, for a similar reason, AB and DC are parallel. Therefore, the figure is a parallelogram.



360. Theorem.—*If, in a quadrilateral, two opposite sides are equal and parallel, the figure is a parallelogram.*

If AD and BC are both equal and parallel, then AB is parallel to DC.

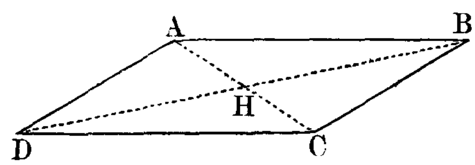
For, joining BD, the triangles thus formed are equal, since they have the side BD common, the side AD equal to BC, and the angle ADB equal to its alternate DBC (284). Hence, the angle ABD is equal to BDC. But these are alternate with reference to the lines AB and DC, and the secant BD.



Therefore, AB and DC are parallel, and the figure is a parallelogram.

361. Theorem.—*The diagonals of a parallelogram bisect each other.*

The diagonals AC and BD are each divided into equal parts at H, the point of intersection.



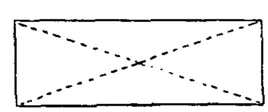
For the triangles ABH and CDH have the sides AB and CD equal (356), the angles ABH and CDH equal (125), and the angles BAH and DCH equal. Therefore, the triangles are equal (285), and AH is equal to CH, and BH to DH.

362. Theorem.—*If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

To be demonstrated by the student.

RECTANGLE

363. If one angle of a parallelogram is right, the others must be right also (354).

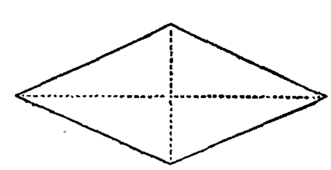


A RECTANGLE is a right angled parallelogram. The rectangle has all the properties of other parallelograms, and the following peculiar to itself, which the student may demonstrate.

364. Theorem.—*The diagonals of a rectangle are equal.*

RHOMBUS.

365. When two adjacent sides of a parallelogram are equal, all its sides must be equal (356).



A RHOMBUS, or, as sometimes called, a LOZENGE, is a parallelogram having all its sides equal.

The rhombus has the following peculiarities, which may be demonstrated by the student.

366. Theorem.—*The diagonals of a rhombus are perpendicular to each other.*

367. Theorem.—*The diagonals of a rhombus bisect its angles.*

SQUARE.

368. A Square is a quadrilateral having its sides equal, and its angles right angles. The square may be shown to have all the properties of the parallelogram (359), of the rectangle, and of the rhombus.

369. Corollary.—The rectangle and the square are the only parallelograms which can be inscribed in a circle (348).

EQUALITY.

370. Theorem.—*Two parallelograms are equal when two adjacent sides and the included angle in the one, are respectively equal to those parts in the other.*

For the remaining sides must be equal (356), and this becomes a case of Article 345.

371. Corollary.—Two rectangles are equal when two adjacent sides of the one, are respectively equal to those parts of the other.

372. Corollary.—Two squares are equal when a side of the one is equal to a side of the other.

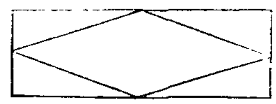
APPLICATIONS.

373. The rectangle is the most frequently used of all quadrilaterals. The walls and floors of our apartments, doors and windows, books, paper, and many other articles, have this form.

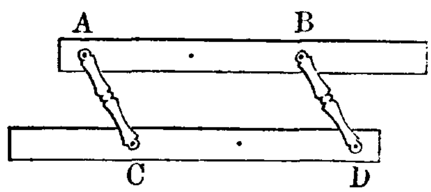
Carpenters make an ingenious use of a geometrical principle in order to make their door and window-frames exactly rectangular. Having made the frame, with its sides equal and its ends equal, they measure the two diagonals, and make the frame take such a shape that these also will be equal.

In this operation, what principle is applied?

374. A rhombus inscribed in a rectangle is the basis of many ornaments used in architecture and other work.



375. An instrument called *parallel rulers*, used in drawing parallel lines, consists of two rulers, connected by cross pieces with pins in their ends. The rulers may turn upon the pins, varying their distance. The distances between the pins along the rulers, that is, AB and CD, must be equal; also, along the cross pieces, that is, AC and BD. Then the rulers will always be parallel to each other. If one ruler be held fast while the other is moved, lines drawn along the edge of the other ruler, at different positions, will be parallel to each other.



What geometrical principles are involved in the use of this instrument?

EXERCISES.

376.—1. State the converse of each theorem that has been given in this chapter, and determine whether each of these converse propositions is true.

2. To construct a parallelogram when two adjacent sides and an angle are given.

3. What parts need be given for the construction of a rectangle?

4. What must be given for the construction of a square?

5. If the four middle points of the sides of any quadrilateral be joined by straight lines, those lines form a parallelogram.

6. If four points be taken, one in each side of a square, at equal distances from the four vertices, the figure formed by joining these successive points is a square.

7. Two parallelograms are similar when they have an angle in the one equal to an angle in the other, and these equal angles included between proportional sides.

MEASURE OF AREA.

377. The standard figure for the measure of surfaces is a square. That is, the unit of area is a square, the side of which is the unit of length, whatever be the extent of the latter.

Other figures might be, and sometimes are, used for this purpose; but the square has been almost universally adopted, because

1. Its form is regular and simple;

2. The two dimensions of the square, its length and breadth, are the same; and,

3. A plane surface can be entirely covered with equal squares.

The truth of the first two reasons is already known to the student: that of the last will appear in the following theorem.

378. Any side of a polygon may be taken as the *base*.

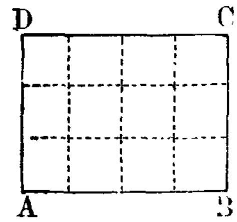
The ALTITUDE of a parallelogram is the distance between the base and the opposite side. Hence, the altitude of a parallelogram may be taken in either of two ways.

AREA OF RECTANGLES.

379. Theorem.—*The area of a rectangle is measured by the product of its base by its altitude.*

That is, if we multiply the number of units of length contained in the base, by the number of those units contained in the altitude, the product is the number of units of area contained in the surface.

Suppose that the base AB and the altitude AD are multiples of the same unit of length, for example, four and three. Divide AB into four equal parts, and through all the points of division extend lines parallel to AD. Divide AD into three equal parts, and through the points of division extend lines parallel to AB.



All the intercepted parts of these two sets of parallels must be equal (357); and all the angles, right angles (124). Thus, the whole rectangle is divided into equal squares (372). The number of these squares is equal to the number in one row multiplied by the number of rows; that is, the number of units of length in the base multiplied by the number in the altitude. In the example taken, this is three times four, or twelve. The result would be the same, whatever the number of divisions in the base and altitude.

If the base and altitude have no common measure, then we may assume the unit of length as small as we please. By taking for the unit a less and less part of the altitude, the base will be made the limit of the lines commensurable with the altitude. Thus, the demonstration is made general.

380. Corollary.—The area of a square is expressed by the second power of the length of its side. Anciently the principles of arithmetic were taught and illustrated by geometry, and we still find the word *square* in common use for the second power of a number.

381. By the method of infinites (203), the latter part of the above demonstration would consist in supposing the base and altitude of the rectangle divided into infinitely small and equal parts; and then proceeding to form infinitesimal squares, as in the former part of the demonstration.

If a straight line move in a direction perpendicular to itself, it describes a rectangle, one of whose dimensions is the given line, and the other is the distance which it has moved. Thus, it appears that the two dimensions which every surface has (33), are combined in the simplest manner in the rectangle.

A rectangle is said to be *contained* by its base and altitude. Thus, also, the area of any figure is called its *superficial contents.*

APPLICATION.

382. All enlightened nations attach great importance to exact and uniform standard measures. In this country the standard of length is a yard measure, carefully preserved by the National Government, at Washington City. By it all the yard measures are regulated.

The standards generally used for the measure of surface, are the square described upon a yard, a foot, a mile, or some other certain length; but the acre, one of the most common measures of surface, is an exception. The number of feet, yards, or rods in one side of a square acre, can only be expressed by the aid of a radical sign.

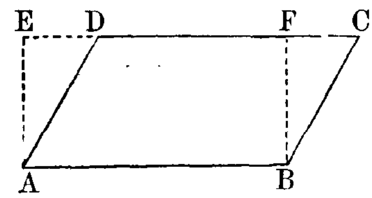
The public lands belonging to the United States are divided into square townships, each containing thirty-six square miles, called *sections.*

AREA OF PARALLELOGRAMS.

383. *The area of a parallelogram is measured by the product of its base by its altitude.*

At the ends of the base AB erect perpendiculars, and produce them till they meet the opposite side, in the points E and F.

Now the right angled triangles AED and BFC are equal, having the side BF equal to AE, since they are perpendiculars between parallels (133); and the side BC equal to AD, by hypothesis (288). If each of these equal triangles be subtracted from the entire figure, ABCE, the remainders ABFE and ABCD must be equivalent. But ABFE is a rectangle having the same base and altitude as the parallelogram ABCD. Hence, the area of the parallelogram is measured by the same product as that which measures the area of the rectangle.



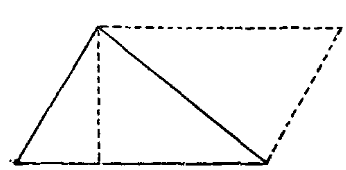
384. Corollary.—Any two parallelograms have their areas in the same ratio as the products of their bases by their altitudes. Parallelograms of equal altitudes have the same ratio as their bases, and parallelograms of equal bases have the same ratio as their altitudes.

385. Corollary.—Two parallelograms are equivalent when they have equal bases and altitudes; or, when the two dimensions of the one are the extremes, and the two dimensions of the other are the means, of a proportion.

AREA OF TRIANGLES.

386. Theorem.—*The area of a triangle is measured by half the product of its base by its altitude.*

For any triangle is one-half of a parallelogram having the same base and altitude (358).



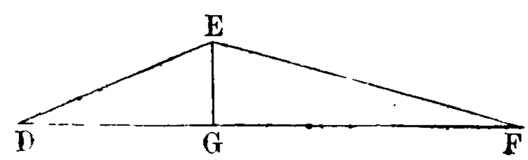
387. Corollary.—The areas of triangles are in the ratio of the products of their bases by their altitudes.

388. Corollary.—Two triangles are equivalent when they have equal bases and altitudes.

389. Corollary.—If a parallelogram and a triangle have equal bases and altitudes, the area of the parallelogram is double that of the triangle.

390. Theorem.—*If from half the sum of the three sides of a triangle each side be subtracted, and if these remainders and the half sum be multiplied together, then the square root of the product will be the area of the triangle.*

Let DEF be any triangle, DF being the base and EG the altitude. Let the extent of the several lines be represented by letters; that is, let



DF $=a$, EF $=b$, DE $=c$, EG $=h$, GF $=m$, DG $=n$, and DE + EF + FD $=s$.

Then (328), $m+n : b+c :: b-c : m-n$.

Therefore, $$m-n=\frac{b^2-c^2}{m+n}.$$

By hypothesis, $$m+n=a.$$

Adding, $$2m=a+\frac{b^2-c^2}{a}.$$

Then, $$m=\frac{a^2+b^2-c^2}{2a}.$$

Again (327), $$m^2+h^2=b^2.$$

Substituting for m^2 its value, and transposing,

$$h^2=b^2-\left(\frac{a^2+b^2-c^2}{2a}\right)^2.$$

Therefore, $$h=\sqrt{b^2-\left(\frac{a^2+b^2-c^2}{2a}\right)^2}.$$

But the area of the triangle is half the product of the base a by the altitude h. Hence,

$$area\ \text{DEF} = \frac{ah}{2} = \frac{a}{2}\sqrt{b^2 - \left(\frac{a^2+b^2-c^2}{2a}\right)^2}.$$

In this expression, we have the area of the triangle in terms of the three sides. For greater facility of calculation it is reduced to the following:

$$area = \tfrac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}.$$

The exact equality of these two expressions is shown by performing as far as is possible the operations indicated in each.

But, by hypothesis, $(a+b+c) = s = 2\left(\frac{s}{2}\right)$.

Therefore, $(a+b-c) = 2\left(\frac{s}{2}-c\right)$,

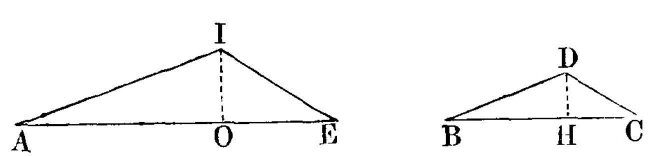
$(a-b+c) = 2\left(\frac{s}{2}-b\right)$,

and, $(-a+b+c) = 2\left(\frac{s}{2}-a\right)$.

Substituting these in the equation of the area, it becomes,

$$area = \sqrt{\left(\frac{s}{2}\right)\left(\frac{s}{2}-a\right)\left(\frac{s}{2}-b\right)\left(\frac{s}{2}-c\right)}.$$

391. Theorem.—*The areas of similar triangles are to each other as the squares of their homologous lines.*



Let AEI and BCD be similar triangles, and IO and DH homologous altitudes.

Then (310), $IO : DH :: AE : BC.$

Multiply by $AE : BC :: AE : BC.$

Then, $AE \times IO : BC \times DH :: \overline{AE}^2 : \overline{BC}^2.$

But (387),

$$AE \times IO : BC \times DH :: \text{area } AEI : \text{area } BCD.$$

Therefore (21),

$$\text{area } AEI : \text{area } BCD :: \overline{AE}^2 : \overline{BC}^2.$$

In a similar manner, prove that the areas have the same ratio as the squares of the altitudes IO and DH, or as the squares of any homologous lines.

AREA OF TRAPEZOIDS.

392. Theorem.—*The area of a trapezoid is equal to half the product of its altitude by the sum of its parallel sides.*

The trapezoid may be divided by a diagonal into two triangles, having for their bases the parallel sides.

The altitude of each of these triangles is equal to that of the trapezoid (264). The area of each triangle being half the product of the common altitude by its base, the area of their sum, or of the whole trapezoid, is half the product of the altitude by the sum of the bases.

EXERCISES.

393.—1. Measure the length and breadth, and find the area of the blackboard; of the floor.

2. To divide a given triangle into any number of equivalent triangles.

3. To divide a given parallelogram into any number of equivalent parallelograms.

4. To divide a given trapezoid into any number of equivalent trapezoids.

5. The area of a triangle is equal to half the product of the perimeter by the radius of the inscribed circle.

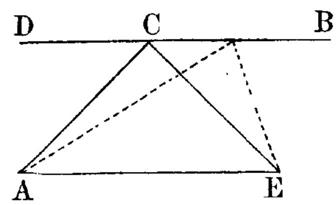
6. What is the radius of the circle inscribed in the triangle whose sides are 8, 10, and 12?

EQUIVALENT SURFACES.

394. ISOPERIMETRICAL figures are those whose perimeters have the same extent.

395. Theorem.—*Of all equivalent triangles of a given base, the one having the least perimeter is isosceles.*

The equivalent triangles having the same base, AE, have also the same altitude (388). Hence, their vertices are in the same line parallel to the base, that is, in DB.



Now, the shortest line that can be made from A to E through some point of DB, will constitute the other two sides of the triangle of least perimeter. This shortest line is the one making equal angles with DB, as ACE, that is, making ACD and ECB equal (115). The angle ACD is equal to its alternate A, and the angle ECB to its alternate E. Therefore, the angles at the base are equal, and the triangle is isosceles.

396. Corollary.—Of all isoperimetrical triangles of a given base, the one having the greatest area is isosceles.

397. To draw a square equivalent to a given figure, is called the *squaring*, or *quadrature* of the figure. How this can be done for any rectilinear figure, is shown in the following.

PROBLEMS IN DRAWING.

398. Problem.—*To draw a rectangle with a given base, equivalent to a given parallelogram.*

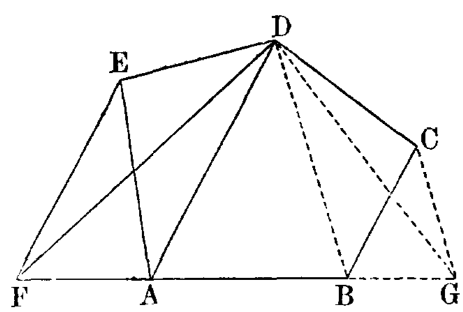
With the given base as a first term, and the base and altitude of the given figure as the second and third terms, find a fourth proportional (319). This is the required altitude (385).

399. Problem.—*To draw a square equivalent to a given parallelogram.*

Find a mean proportional between the base and altitude of the given figure (330). This is the side of the square (385).

400. Problem.—*To draw a triangle equivalent to a given polygon.*

Let ABCDE be the given polygon. Join DA. Produce BA, and through E draw EF parallel to DA. Join DF.



Now, the triangles DAF and DAE are equivalent, for they have the same base DA, and equal altitudes, since their vertices are in the line EF parallel to the base (264). To each of these equals, add the figure ABCD, and we have the quadrilateral FBCD equivalent to the polygon ABCDE. In this manner, the number of sides may be diminished till a triangle is formed equivalent to the given polygon. In this diagram it is the triangle FDG.

401. Problem.—*To draw a square equivalent to a given triangle.*

Find a mean proportional between the altitude and half the base of the triangle. This will be the side of the required square.

EQUIVALENT SQUARES.

402. Having shown (379) how an area is expressed by the product of two lengths, it follows that an equation will represent equivalent surfaces, if each of its terms is composed of two factors which represent lengths.

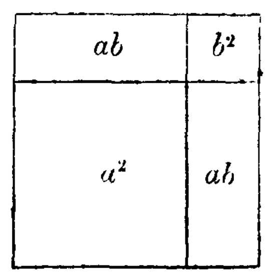
For example, let a and b represent the lengths of two straight lines. Now we know, from algebra, that whatever be the value of a and b,

$$(a+b)^2 = a^2 + 2ab + b^2.$$

This formula, therefore, includes the following geometrical

403. Theorem.—*The square described upon the sum of two lines is equivalent to the sum of the squares described on the two lines, increased by twice the rectangle contained by these two lines.*

Since the truths of algebra are universal in their application, this theorem is demonstrated by the truth of the above equation.



Such a proof is called *algebraic*. It is also called *analytical*, but with doubtful propriety.

Let the student demonstrate the theorem geometrically, by the aid of this diagram.

404. Theorem.—*The square described on the difference of two straight lines is equivalent to the sum of the squares described on the two lines, diminished by twice the rectangle contained by those lines.*

This is a consequence of the truth of the equation,

$$(a-b)^2 = a^2 - 2ab + b^2.$$

405. Theorem.—*The rectangle contained by the sum and the difference of two straight lines is equivalent to the difference of the squares of those lines.*

This, again, is proved by the principle expressed in the equation,

$$(a+b)\,(a-b)=a^2-b^2.$$

406. These two theorems may also be demonstrated by purely geometrical reasoning.

The algebraic method is sometimes called the modern, while the other is called the ancient geometry. The algebraic method was invented by Descartes, in the seventeenth century, while the other is twenty centuries older.

THE PYTHAGOREAN THEOREM.

407. Since numerical equations represent geometrical truths, the following theorem might be inferred from Article 327.

This is called the *Pythagorean* Theorem, because it was discovered by Pythagoras. It is also known as the *Forty-seventh* Proposition, that being its number in the First Book of Euclid's Elements.

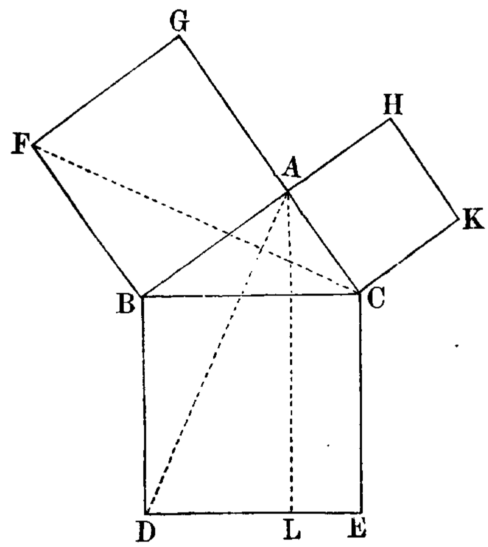
It has been demonstrated in a great variety of ways. One is by dividing the three squares into parts, so that the several parts of the large square are respectively equal to the several parts of the two others.

The fame of this theorem makes it proper to give here the demonstration from Euclid.

408. Theorem.—*The square described on the hypotenuse of a right angled triangle is equivalent to the sum of the squares described on the two legs.*

Let ABC be a right angled triangle, having the right angle BAC. The square described on the side BC is equivalent to the sum of the two squares described on BA and AC. Through A make AL parallel to BD, and join AD and FC.

Then, because each of the angles BAC and BAG is a right angle, the line GAC is one straight line (100). For the same reason, BAH is one straight line.



The angles FBC and DBA are equal, since each is the sum of a right angle and the angle ABC. The two triangles FBC and DBA are equal, for the side FB in the one is equal to BA in the other, and the side BC in the one is equal to BD in the other, and the included angles are equal, as just proved.

Now, the area of the parallelogram BL is double that of the triangle DBA, because they have the same base BD, and the same altitude DL (389). And the area of the square BG is double that of the triangle FBC, because these also have the same base BF, and the same altitude FG. But doubles of equals are equal (7). Therefore, the parallelogram BL and the square BG are equivalent.

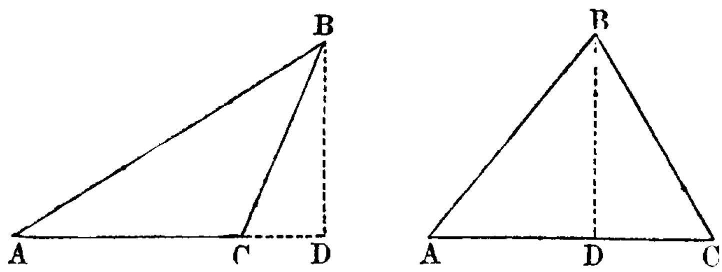
In the same manner, by joining AE and BK, it is demonstrated that the parallelogram CL and the square CH are equivalent. Therefore, the whole square BE, described on the hypotenuse, is equivalent to the two squares BG and CH, described on the legs of the right angled triangle.

409. Corollary.—The square described on one leg is equivalent to the difference of the squares on the hypotenuse and the other leg.

410. If from the extremities of one line perpendiculars be let fall upon another, then the part of the second line between the perpendiculars is called the *projection* of the first line on the second. If one end of the first line is in the second, then only one perpendicular is necessary.

411. Theorem.—*The square described on the side opposite to an acute angle of a triangle, is equivalent to the sum of the squares described on the other two sides, diminished by twice the rectangle contained by one of these sides and the projection of the other on that side.*

Let A be the acute angle, and from B let a perpendicular fall upon AC, produced if necessary. Then,



AD is the projection of AB upon AC. And it is to be proved that the square on BC is equivalent to the sum of the squares on AB and on AC, diminished by twice the rectangle contained by AC and AD.

For (409), $\overline{BD}^2 = \overline{AB}^2 - \overline{AD}^2$;

and (404), $\overline{CD}^2 = \overline{AC}^2 + \overline{AD}^2 - 2AC \times AD.$

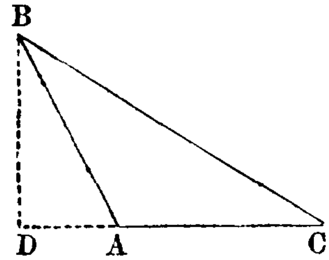
By addition, $\overline{BD}^2 + \overline{CD}^2 = \overline{AB}^2 + \overline{AC}^2 - 2AC \times AD.$

But the square on BC is equivalent to $\overline{BD}^2 + \overline{CD}^2$ (408). Therefore, it is also equivalent to

$$\overline{AB}^2 + \overline{AC}^2 - 2AC \times AD.$$

412. Theorem.—*The square described on the side opposite an obtuse angle of a triangle, is equivalent to the sum of the squares described on the other two sides, increased by twice the rectangle of one of those sides and the projection of the other on that side.*

In the triangle ABC, the square on BC which is opposite the obtuse angle at A, is equivalent to the sum of the squares on AB and on AC, and twice the rectangle contained by CA and AD.



For, $\overline{BD}^2 = \overline{AB}^2 - \overline{AD}^2$;

and (403), $\overline{CD}^2 = \overline{AC}^2 + \overline{AD}^2 + 2AC \times AD.$

By addition, $\overline{BD}^2 + \overline{CD}^2 = \overline{AB}^2 + \overline{AC}^2 + 2AC \times AD.$

But, $\overline{BC}^2 = \overline{BD}^2 + \overline{CD}^2.$

Therefore, $\overline{BC}^2$ is equivalent to

$$\overline{AB}^2 + \overline{AC}^2 + 2AC \times AD.$$

413. Corollary.—If the square described on one side of a triangle is equivalent to the sum of the squares described on the other two sides, then the opposite angle is a right angle. For the last two theorems show that it can be neither acute nor obtuse.

EXERCISES.

414.—1. When a quadrilateral has its opposite angles supplementary, a circle can be circumscribed about it.

2. From a given isosceles triangle, to cut off a trapezoid which

shall have the same base as the triangle, and the remaining three sides equal to each other.

3. The lines which bisect the angles of a parallelogram, form a rectangle whose diagonals are parallel to the sides of the parallelogram.

4. In any parallelogram, the distance of one vertex from a straight line passing through the opposite vertex, is equal to the sum or difference of the distances of the line from the other two vertices, according as the line is without or within the parallelogram.

5. When one diagonal of a quadrilateral divides the figure into equal triangles, is the figure necessarily a parallelogram?

6. Demonstrate the theorem, Article 329, by Articles 113 and 387.

7. What is the area of a lot, which has the shape of a right angled triangle, the longest side being 100 yards, and one of the other sides 36 yards.

8. Can every triangle be divided into two equal parts? Into three? Into nine?

9. Two parallelograms having the same base and altitude are equivalent.

To be demonstrated without using Articles 379 or 383.

10. A triangle is divided into two equivalent parts, by a line from the vertex to the middle of the base.

To be demonstrated without the aid of the principles of this chapter.

11. To divide a triangle into two equivalent parts, by a line drawn from a given point in one of the sides.

12. Of all equivalent parallelograms having equal bases, what one has the minimum perimeter?

13. Find the locus of the points such that the sum of the squares of the distances of each from two given points, shall be equivalent to the square of the line joining the given points.

CHAPTER VII.

POLYGONS.

415. Hitherto the student's attention has been given to polygons of three and of four sides only. He has seen how the theories of similarity and of linear ratio have grown out of the consideration of triangles; and how the study of quadrilaterals gives us the principles for the measure of surfaces, and the theory of equivalent figures.

In the present chapter, some principles of polygons of any number of sides will be established.

A *Pentagon* is a polygon of five sides; a *Hexagon* has six sides; an *Octagon*, eight; a *Decagon*, ten; a *Dodecagon*, twelve; and a *Pentedecagon*, fifteen.

The following propositions on diagonals, and on the sum of the angles, are more general statements of those in Articles 340 to 346.

DIAGONALS.

416. Theorem.—*The number of diagonals from any vertex of a polygon, is three less than the number of sides.*

For, from each vertex a diagonal may extend to every other vertex except itself, and the one adjacent on each side. Thus, the number is three less than the number of vertices, or of sides.

417. Corollary.—The diagonals from one vertex divide a polygon into as many triangles as the polygon has sides, less two.

Polygons may be divided into this number, or into a greater number of triangles, in various ways; but a polygon can not be divided into a less number of triangles than here stated.

418. Corollary.—The whole number of diagonals possible in a polygon of n sides, is $\frac{1}{2} n (n-3)$. For, if we count the diagonals at all the n vertices, we have $n (n-3)$, but this is counting each diagonal at both ends. This last product must therefore be divided by two.

EQUAL POLYGONS.

419. Theorem.—*Two polygons are equal when they are composed of the same number of triangles respectively equal and similarly arranged.*

This is an immediate consequence of the definition of equality (40).

420. Corollary.—Conversely, two equal polygons may be divided into the same number of triangles respectively equal and similarly arranged.

421. Theorem.—*Two polygons are equal when all the sides and all the diagonals from one vertex of the one, are respectively equal to the same lines in the other, and are similarly arranged.*

For each triangle in the one would have its three sides equal to the similarly situated triangle in the other, and would be equal to it (282). Therefore, the polygons would be equal (419).

422. Theorem—*Two polygons are equal when all the sides and the angles of the one are respectively equal to the same parts of the other, and are similarly arranged.*

For each triangle in the one is equal to its homologous triangle in the other, since they have two sides and the included angle equal.

It is enough for the hypothesis of this theorem, that all the angles except three be among the equal parts.

SUM OF THE ANGLES.

423. Theorem.—*The sum of all the angles of a polygon is equal to twice as many right angles as the polygon has sides, less two.*

For the polygon may be divided into as many triangles as it has sides, less two (417); and the angles of these triangles coincide altogether with those of the polygon.

The sum of the angles of each triangle is two right angles. Therefore, the sum of the angles of the polygon is equal to twice as many right angles as it has sides, less two.

The remark in Article 346 applies as well to this theorem.

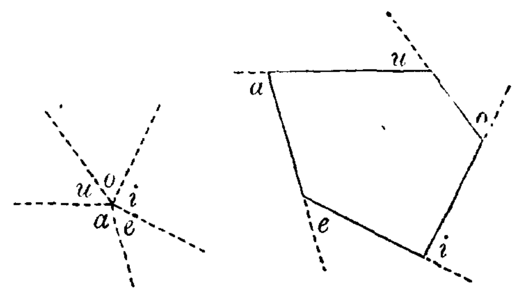
424. Let R represent a right angle; then the sum of the angles of a polygon of n sides is $2\ (n-2)\ R$; or, it may be written thus, $(2n-4)\ R$.

The student should illustrate each of the last five theorems with one or more diagrams.

425. Theorem.—*If each side of a convex polygon be produced, the sum of all the exterior angles is equal to four right angles.*

Let the sides be produced all in one way; that is, all to the right or all to the left. Then, from any point in the plane, extend lines parallel to the sides thus produced, and in the same directions.

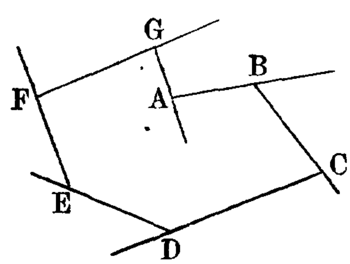
The angles thus formed are equal in number to the exterior angles of the polygon, and are respectively equal to them (138). But the sum of those formed about the point is equal to four right angles (92).



Therefore, the sum of the exterior angles of the polygon is equal to four right angles.

426. This theorem will also be true of concave polygons, if the angle formed by producing one side of the reëntrant angle is considered as a negative quantity.

Thus, the remainder, after subtracting the angle formed at A by producing GA, from the sum of the angles formed at B, C, D, E, F, and G, is four right angles. This may be demonstrated by the aid of the previous theorem (423).



EXERCISES.

427.—1. What is the number of diagonals that can be in a pentagon? In a decagon?

2. What is the sum of the angles of a hexagon? Of a dodecagon?

3. What is the greatest number of acute angles which a convex polygon can have?

4. Join any point within a given polygon with every vertex of the polygon, and with the figure thus formed, demonstrate the theorem, Article 423.

5. Demonstrate the theorem, Article 425, by means of Article 423, and without using Article 92.

PROBLEMS IN DRAWING.

428. Problem.—*To draw a polygon equal to a given polygon.*

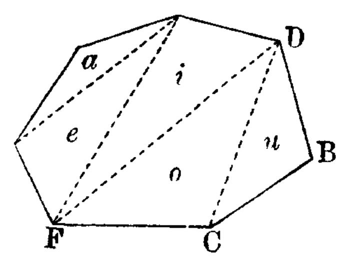
By diagonals divide the given polygon into triangles. The problem then consists in drawing triangles equal to given triangles.

429. Problem.—*To draw a polygon when all its sides and all the diagonals from one vertex, are given in their proper order.*

This consists in drawing triangles with sides equal to three given lines (295).

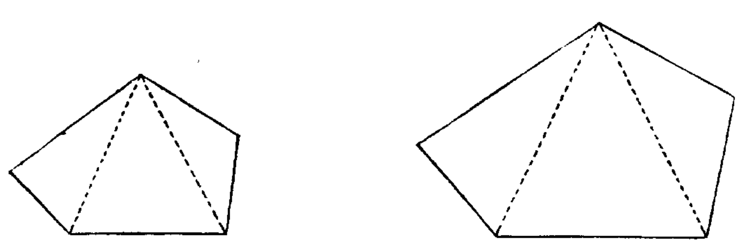
430. Problem.—*To draw a polygon when the sides and angles are given in their order.*

It is enough for this problem if all the angles except three be given. For, suppose first that the angles not given are consecutive, as at D, B, and C. Then, draw the triangles *a*, *e*, *i*, and *o* (297). Then, having DC, complete the polygon by drawing the triangle DBC from its three known sides (295). Suppose the angles not given were D, C, and F. Then, draw the triangles *a*, *e*, and *i*, and separately, the triangle *u*. Then, having the three sides of the triangle *o*, it may be drawn, and the polygon completed.



SIMILAR POLYGONS.

431. Theorem.—*Similar polygons are composed of the same number of triangles, respectively similar and similarly arranged.*



Since the figures are similar, every angle in one has its corresponding equal angle in the other (303). If, then, diagonals be made to divide one of the polygons into triangles, every angle thus formed may have its corresponding equal angle in the other. Therefore, the triangles of one polygon are respectively similar to those of the other, and are similarly arranged.

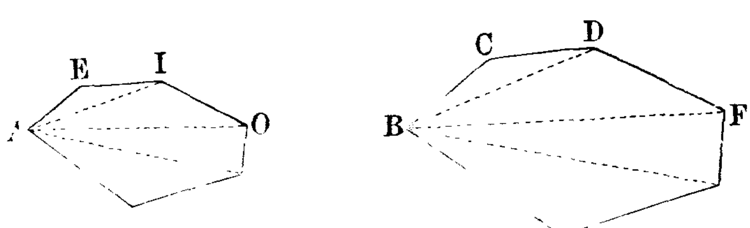
432. Theorem.—*If two polygons are composed of the same number of triangles which are respectively similar and are similarly arranged, the polygons are similar.*

By the hypothesis, all the angles formed by the given lines in one polygon have their corresponding equal angles in the other. It remains to be proved that angles formed by any other lines in the one have their corresponding equal angles in the other polygon.

This may be shown by reasoning, in the same manner as in the case of triangles (304). Let the student make the diagrams and complete the demonstration.

433. Theorem.—*Two polygons are similar when the angles formed by the sides are respectively equal, and there is the same ratio between each side of the one and its homologous side of the other.*

Let all the diagonals possible extend from a vertex A



of one polygon, and the same from the homologous vertex B of the other polygon.

Now the triangles AEI and BCD are similar, because they have two sides proportional, and the included angles equal (317).

Therefore, EI : CD : : AI : BD.

But, by hypothesis, EI : CD : : IO : DF.

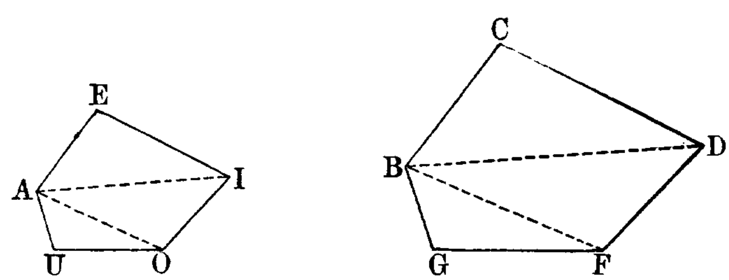
Then (21), AI : BD : : IO : DF.

Also, if we subtract the equal angles EIA and CDB from the equal angles EIO and CDF, the remainders AIO and BDF are equal. Hence, the triangles AIO and BDF are similar. In the same manner, prove that each of the triangles of the first polygon is similar to its corresponding triangle in the other. Therefore, the figures are similar (432).

As in the case of equal polygons (422 and 430), it is only necessary to the hypothesis of this proposition, that all the angles except three in one polygon be equal to the homologous angles in the other.

434. Theorem.—*In similar polygons the ratio of two homologous lines is the same as of any other two homologous lines.*

For, since the polygons are similar, the triangles which



compose them are also similar, and (309),

AE : BC : : EI : CD : : AI : BD : : IO : DF, etc.

This common ratio is the linear ratio of the two figures.

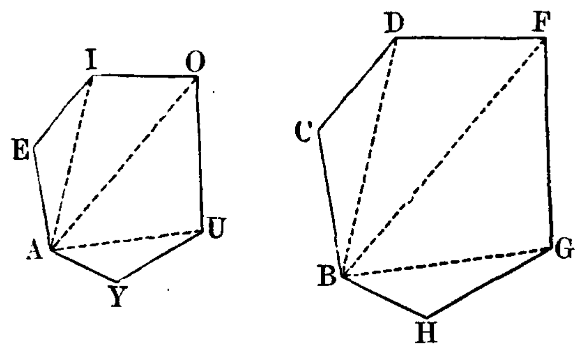
Let the student show that the perpendicular let fall from E upon OU, and the homologous line in the other polygon, have the linear ratio of the two figures.

435. Theorem.—*The perimeters of similar polygons are to each other as any two homologous lines.*

The student may demonstrate this theorem in the same manner as the corresponding propositions in triangles (312).

436. Theorem.—*The area of any polygon is to the area of a similar polygon, as the square on any line of the first is to the square on the homologous line of the second.*

Let the polygons BCD, etc., and AEI, etc., be divided into triangles by homologous diagonals. The triangles thus formed in the one are similar to those formed in the other (431).



Therefore (391),

area BCD : area AEI : : $\overline{BD}^2$: $\overline{AI}^2$: : area BDF : area AIO : : $\overline{BF}^2$: $\overline{AO}^2$: : area BFG : area AOU : : $\overline{BG}^2$: $\overline{AU}^2$: : area BGH : area AUY.

Selecting from these equal ratios the triangles, area BCD : area AEI : : area BDF : area AIO : : area BFG : area AOU : : area BGH : area AUY.

Therefore (23), area BCDFGHB : area AEIOUYA : : area BCD : area AEI; or, as $\overline{BC}^2$: $\overline{AE}^2$; or, as the areas of any other homologous parts; or, as the squares of any other homologous lines.

437. Corollary.—The superficial ratio of two similar polygons is always the second power of their linear ratio.

EXERCISES.

438.—1. Compose two polygons of the same number of triangles respectively similar, but not similarly arranged.

2. To draw a triangle similar to a given triangle, but with double the area.

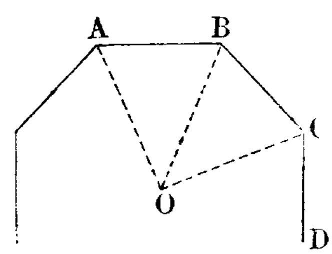
3. What is the relation between the areas of the equilateral triangles described on the three sides of a right angled triangle?

REGULAR POLYGONS.

439. A REGULAR POLYGON is one which has all its sides equal, and all its angles equal. The square and the equilateral triangle are regular polygons.

440. Theorem.—*Within a regular polygon there is a point equally distant from the vertices of all the angles.*

Let ABCD, etc., be a regular polygon, and let lines bisecting the angles A and B extend till they meet at O. These lines will meet, for the interior angles which they make with AB are both acute (137).



In the triangle ABO, the angles at A and B are equal, being halves of the equal angles of the polygon. Therefore, the opposite sides AO and BO are equal (275).

Join OC. Now, the triangles ABO and BCO are equal, for they have the side AO of the first equal to BO of the second, the side AB equal to BC, because the polygon is regular, and the included angles OAB and OBC equal, since they are halves of angles of the polygon. Hence, BO is equal to OC.

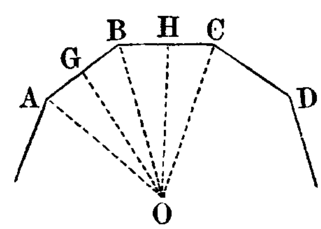
Then, the angle OCB is equal to OBC (268), and OC bisects the angle BCD, which is equal to ABC. In the same manner, it is proved that OC is equal to OD, and so on. Therefore, the point O is equally distant from all the vertices.

CIRCUMSCRIBED AND INSCRIBED.

441. Corollary.—Every regular polygon may have a circle circumscribed about it. For, with O as a center and OA as a radius, a circumference may be described passing through all the vertices of the polygon (153).

442. Theorem.—*The point which is equally distant from the vertices is also equally distant from the sides of a regular polygon.*

The triangles OAB, OBC, etc., are all isosceles. If perpendiculars be let fall from O upon the several sides AB, BC, etc., these sides will be bisected (271). Then, the perpendiculars will be equal, for they will be sides of equal triangles. But they measure the distances from O to the several sides of the polygon. Therefore, the point O is equally distant from all the sides of the polygon.



443. Corollary.—Every regular polygon may have a circle inscribed in it. For with O as a center and OG as a radius, a circumference may be described passing through the feet of all these perpendiculars, and tangent to all the sides of the polygon (178), and therefore inscribed in it (253).

444. Corollary.—A regular polygon is a symmetrical figure.

445. The center of the circumscribed or inscribed circle is also called the *center of a regular polygon.* The radius of the circumscribed circle is also called the *radius of a regular polygon.*

The APOTHEM of a regular polygon is the radius of the inscribed circle.

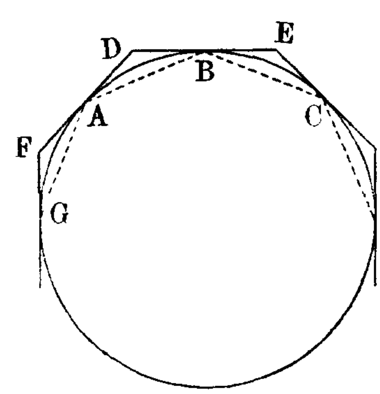
446. Theorem.—*If the circumference of a circle be divided into equal arcs, the chords of those equal arcs will be the sides of a regular polygon.*

For the sides are all equal, being the chords of equal arcs (185); and the angles are all equal, being inscribed in equal arcs (224).

447. Corollary.—An angle formed at the center of a regular polygon by lines from adjacent vertices, is an aliquot part of four right angles, being the quotient of four right angles divided by the number of the sides of the polygon.

448. Theorem.—*If a circumference be divided into equal arcs, and lines tangent at the several points of division be produced until they meet, these tangents are the sides of a regular polygon.*

Let A, B, C, etc., be points of division, and F, D, and E points where the tangents meet.



Join GA, AB, and BC.

Now, the triangles GAF, ABD, and BCE have the sides GA, AB, and BC equal, as they are chords of equal arcs; and the angles at G, A, B, and C equal, for each is formed by a tangent and chord which intercept equal arcs (226). Therefore, these triangles are all isosceles (275), and all equal (285); and the angles F, D, and E are equal. Also, FD and DE, being doubles of equals, are equal. In the same manner, it is proved that all the angles of the polygon FDE, etc., are equal, and that all its sides are equal. Therefore, it is a regular polygon.

REGULAR POLYGONS SIMILAR.

449. Theorem.—*Regular polygons of the same number of sides are similar.*

Since the polygons have the same number of sides, the sum of all the angles of the one is equal to the sum of all the angles of the other (423). But all the angles of a regular polygon are equal (439). Dividing the equal sums by the number of angles (7), it follows that an angle of the one polygon is equal to an angle of the other.

Again: all the sides of a regular polygon are equal. Hence, there is the same ratio between a side of the first and a side of the second, as between any other side of the first and a corresponding side of the second. Therefore, the polygons are similar (433).

450. Corollary.—The areas of two regular polygons of the same number of sides are to each other as the squares of their homologous lines (436).

451. Corollary.—The ratio of the radius to the side of a regular polygon of a given number of sides, is a constant quantity. For a radius of one is to a radius of any other, as a side of the one is to a side of the other (434). Then, by alternation (19), the radius is to the side of one regular polygon, as the radius is to the side of any other regular polygon of the same number of sides.

452. Corollary.—The same is true of the apothem and side, or of the apothem and radius.

PROBLEMS IN DRAWING.

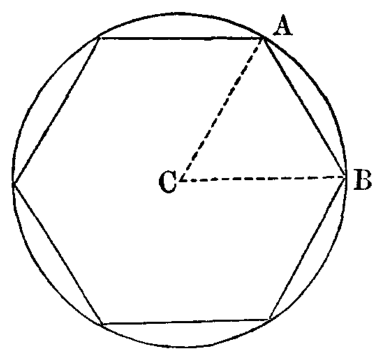
453. Problem.—*To inscribe a square in a given circle.*

Draw two diameters perpendicular to each other. Join their extremities by chords. These chords form an inscribed square.

For the angles at the center are equal by construction (90). Therefore, their intercepted arcs are equal (197), and the chords of those arcs are the sides of a regular polygon (446).

454. Problem.—*To inscribe a regular hexagon in a circle.*

Suppose the problem solved and the figure completed. Join two adjacent angles with the center, making the triangle ABC.



Now, the angle C, being measured by one-sixth of the circumference, is equal to one-sixth of four right angles, or one-third of two right angles. Hence, the sum of the two angles, CAB and CBA, is two-thirds of two right angles (256). But CA and CB are equal, being radii; therefore, the angles CAB and CBA are equal (268), and each of them must be one-third of two right angles. Then, the triangle ABC, being equiangular, is equilateral (276). Therefore, the side of an inscribed regular hexagon is equal to the radius of the circle.

The solution of the problem is now evident—apply the radius to the circumference six times as a chord.

455. Corollary.—Joining the alternate vertices makes an inscribed equilateral triangle.

456. Problem.—*To inscribe a regular decagon in a given circle.*

Divide the radius CA in extreme and mean ratio, at the point B. Then BC is equal to the side of a regular inscribed decagon. That is, if we apply BC as a chord, its arc will be one-tenth of the whole circumference.

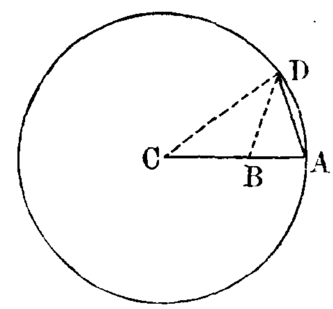
Take AD, making the chord AD equal to BC. Then join DC and DB.

Then, by construction, CA : CB : : CB : BA.

Substituting for CB its equal DA,

CA : DA : : DA : BA.

Then the triangles CDA and BDA are similar, for they have those sides proportional which include the common angle A (317). But the triangle CDA being isosceles, the triangle BDA is the same. Hence, DB is equal to DA, and also to BC. Therefore, the angle C is equal to the angle BDC (268). But it is also equal to BDA. It follows that the angle CDA is twice the angle C. The angle at A being equal to CDA, the angle C must be one-fifth of the sum of these three angles; that is, one-fifth of two right angles (255), or one-tenth of four right angles. Therefore, the arc AD is one-tenth of the circumference (207); and the chord AD is equal to the side of an inscribed regular decagon.



457.—Corollary.—By joining the alternate vertices of a decagon, we may inscribe a regular pentagon.

458. Corollary.—A regular pentedecagon, or polygon of fifteen sides, may be inscribed, by subtracting the arc subtended by the side of a regular decagon from the arc subtended by the side of a regular hexagon. The remainder is one-fifteenth of the circumference, for $\frac{1}{6}-\frac{1}{10}=\frac{1}{15}$.

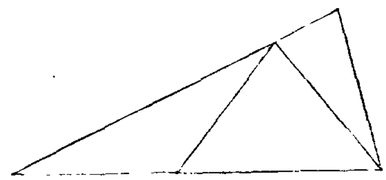
459. Problem.—*Given a regular polygon inscribed in a circle, to inscribe a regular polygon of double the number of sides.*

Divide each arc subtended by a given side into two equal parts (194). Join the successive points into which the circumference is divided. The figure thus formed is the required polygon.

460. We have now learned how to inscribe regular polygons of 3, 4, 5, and 15 sides, and of any number that may arise from doubling either of these four.

The problem, to inscribe a regular polygon in a circle by means of straight lines and arcs of circles, can be solved in only a limited number of cases. It is evident that the solution depends upon the division of the circumference into any number of equal parts; and this depends upon the division of the sum of four right angles into aliquot parts.

461. Notice that the regular decagon was drawn by the aid of two isosceles triangles composing a third, one of the two being similar to the whole. Now, if we could combine three isosceles triangles in this manner, we could draw a regular polygon of fourteen, and then one of seven sides.



However, this can not be done by means only of straight lines and arcs of circles.

The regular polygon of seventeen sides has been drawn in more than one way, using only straight lines and arcs of circles. It has also been shown, that by the same means a regular polygon of two hundred and fifty-seven sides may be drawn. No others are known where the number of the sides is a prime number.

462. Problem.—*Given a regular polygon inscribed in a circle, to circumscribe a similar polygon.*

The vertices of the given polygon divide the circumference into equal parts. Through these points draw tangents. These tangents produced till they meet, form the required polygon (448).

EXERCISES.

463.—1. First in right angles, and then in degrees, express the value of an angle of each regular polygon, from three sides up to twenty.

2. First in right angles, and then in degrees, express the value of an angle at the center, subtended by one side of each of the same polygons.

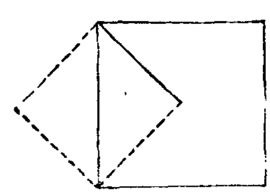
3. To construct a regular octagon of a given side.

4. To circumscribe a circle about a regular polygon.

5. To inscribe a circle in a regular polygon.

6. Given a regular inscribed polygon, to circumscribe a similar polygon whose sides are parallel to the former.

7. The diagonal of a square is to its side as the square root of 2 is to 1.

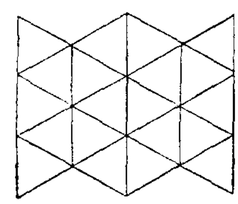


A PLANE OF REGULAR POLYGONS.

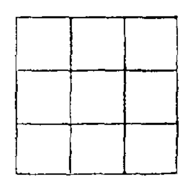
464. In order that any plane surface may be entirely covered by equal polygons, it is necessary that the figures be such, and such only, that the sum of three or more of their angles is equal to four right angles (92).

Hence, to find what regular polygons will fit together so as to cover any plane surface, take them in order according to the number of their sides.

Each angle of an equilateral triangle is equal to one-third of two right angles. Therefore, six such angles exactly make up four right angles; and the equilateral triangle is such a figure as is required.

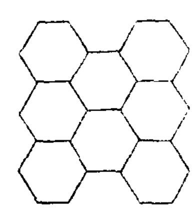


465. Each angle of the square is a right angle, four of which make four right angles. So that a plane can be covered by equal squares.



One angle of a regular pentagon is the fifth part of six right angles. Three of these are less than, and four exceed four right angles; so that the regular pentagon is not such a figure as is required.

466. Each angle of a regular hexagon is one-sixth of eight right angles. Three such make up four right angles. Hence, a plane may be covered with equal regular hexagons. This combination is remarkable as being the one adopted by bees in forming the honeycomb.



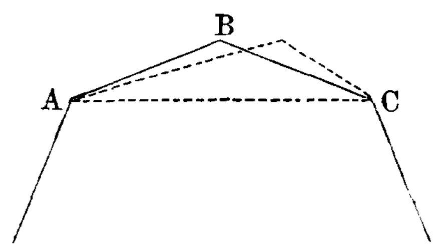
467. Since each angle of a regular polygon evidently increases when the number of sides increases, and since three angles of a regular hexagon are equal to four right angles, therefore, three angles of any regular polygon of more than six sides, must exceed four right angles.

Hence, no other regular figures exist for the purpose here required, except the equilateral triangle, the square, and the regular hexagon.

ISOPERIMETRY.

468. Theorem.—*Of all equivalent polygons of the same number of sides, the one having the least perimeter is regular.*

Of several equivalent polygons, suppose AB and BC to be two adjacent sides of the one having the least perimeter. It is to be proved, first, that these sides are equal.

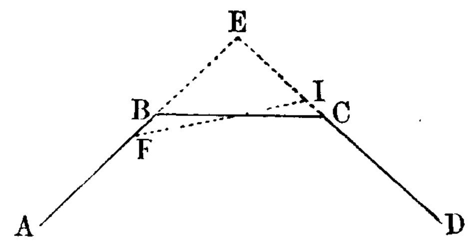


Join AC. Now, if AB and BC were not equal, there could be constructed on the base AC an isosceles triangle equivalent to ABC, whose sides would have less extent (395). Then, this new triangle, with the rest of the polygon, would be equivalent to the given polygon, and have a less perimeter, which is contrary to the hypothesis.

It follows that AB and BC must be equal. So of every two adjacent sides. Therefore, the polygon is equilateral.

It remains to be proved that the polygon will have all its angles equal.

Suppose AB, BC, and CD to be adjacent sides. Produce AB and CD till they meet at E. Now the triangle BCE is isosceles. For if EC, for example, were longer than EB, we could then take EI equal to EB, and EF equal to EC, and we could join FI, making the two triangles EBC and EIF equal (284).



Then, the new polygon, having AFID for part of its perimeter, would be equivalent and isoperimetrical to the given polygon having ABCD as part of its perimeter. But the given polygon has, by hypothesis, the least possible perimeter, and, as just proved, its sides AB, BC, and CD are equal.

If the new polygon has the same area and perimeter, its sides also, for the same reason, must be equal; that is, AF, FI, and ID. But this is absurd, for AF is less than AB, and ID is greater than CD. Therefore, the supposition that EC is greater than EB, which supposition led to this conclusion, is false. Hence, EB and EC must be equal.

Therefore, the angles EBC and ECB are equal (268), and their supplements ABC and BCD are equal. Thus, it may be shown that every two adjacent angles are equal.

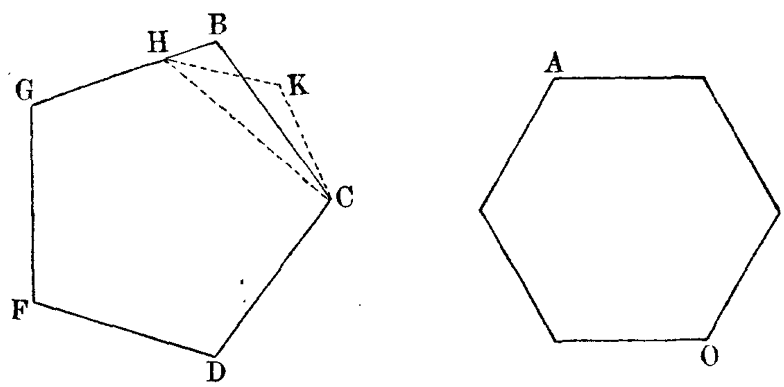
It being proved that the polygon has its sides equal and its angles equal, it is regular.

469. Corollary.—Of all isoperimetrical polygons of the same number of sides, that which is regular has the greatest area.

470. Theorem.—*Of all regular equivalent polygons, that which has the greatest number of sides has the least perimeter.*

It will be sufficient to demonstrate the principle, when one of the equivalent polygons has one side more than the other.

In the polygon having the less number of sides, join the vertex C to any point, as H, of the side BG. Then,



on CH construct an isosceles triangle, CKH, equivalent to CBH.

Then HK and KC are less than HB and BC; therefore, the perimeter GHKCDF is less than the perimeter of its equivalent polygon GBCDF. But the perimeter of the regular polygon AO is less than the perimeter of its equivalent irregular polygon of the same number of sides, GHKCDF (468). So much more is it less than the perimeter of GBCDF.

471. Corollary.—Of two regular isoperimetrical polygons, the greater is that which has the greater number of sides.

EXERCISES.

472.—1. Find the ratios between the side, the radius, and the apothem, of the regular polygons of three, four, five, six, and eight sides.

2. If from any point within a given regular polygon, perpendiculars be let fall on all the sides, the sum of these perpendiculars is a constant quantity.

3. If from all the vertices of a regular polygon, perpendiculars be let fall on a straight line which passes through its center, the

sum of the perpendiculars on one side of this line is equal to the sum of those on the other.

4. If a regular pentagon, hexagon, and decagon be inscribed in a circle, a triangle having its sides respectively equal to the sides of these three polygons will be right angled.

5. If two diagonals of a regular pentagon cut each other, each is divided in extreme and mean ratio.

6. Three houses are built with walls of the same aggregate length; the first in the shape of a square, the second of a rectangle, and the third of a regular octagon. Which has the greatest amount of room, and which the least?

7. Of all triangles having two sides respectively equal to two given lines, the greatest is that where the angle included between the given sides is a right angle.

8. In order to cover a pavement with equal blocks, in the shape of regular polygons of a given area, of what shape must they be that the entire extent of the lines between the blocks shall be a minimum.

9. All the diagonals being formed in a regular pentagon, the figure inclosed by them is a regular pentagon.

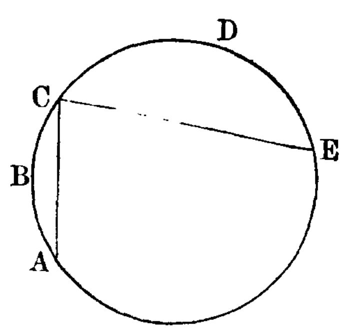
CHAPTER VIII.

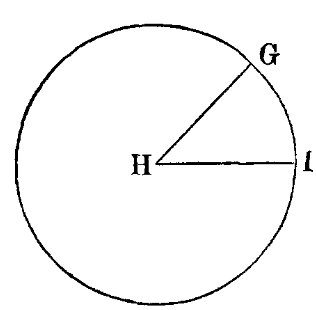
CIRCLES.

473. The properties of the curve which bounds a circle, and of some straight lines connected with it, were discussed in a former chapter. Having now learned the properties of polygons, or rectilinear figures inclosing a plane surface, the student is prepared for the study of the circle as a figure inclosing a surface.

The circle is the only curvilinear figure treated of in Elementary Geometry. Its discussion will complete this portion of the work. The properties of other curves, such as the ellipse which is the figure of the orbits of the planets, are usually investigated by the application of algebra to geometry.

474. A SEGMENT of a circle is that portion cut off by a secant or a chord. Thus, ABC and CDE are segments.





A SECTOR of a circle is that portion included between two radii and the arc intercepted by them. Thus, GHI is a sector.

THE LIMIT OF INSCRIBED POLYGONS.

475. Theorem.—*A circle is the limit of the polygons which can be inscribed in it, also of those which can be circumscribed about it.*

Having a polygon inscribed in a circle, a second polygon may be inscribed of double the number of sides. Then, a polygon of double the number of sides of the second may be inscribed, and the process repeated at will.

Let the student draw a diagram, beginning with an inscribed square or equilateral triangle. Very soon the many sides of the polygon become confused with the circumference. Suppose we begin with a circumscribed regular polygon; here, also, we may circumscribe a regular polygon of double the number of sides. By repeating the process a few times, the polygon becomes inseparable from the circumference.

The mental process is not subject to the same limits that we meet with in drawing the diagrams. We may conceive the number of sides to go on increasing to any number whatever. At each step the inscribed polygon grows larger and the circumscribed grows smaller, both becoming more nearly identical with the circle.

Now, it is evident that by the process described, the polygons can be made to approach as nearly as we please to equality with the circle (35 and 36), but can never entirely reach it. The circle is therefore the limit of the polygons (198).

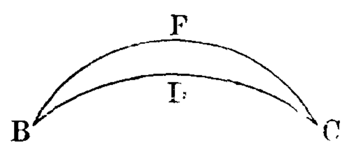
476. Corollary.—A circle is the limit of all regular polygons whose radii are equal to its radius. It is also the limit of all regular polygons whose apothems are equal to its radius. The circumference is the limit of the perimeters of those polygons.

477. By the method of infinites, the circle is considered as a regular polygon of an infinite number of sides, each side being an infinitesimal straight line. But the method of limits is preferred in this place, because, strictly speaking, the circle is not a polygon, and the circumference is not a broken line.

The above theorem establishes only this, that whatever is true of all inscribed, or of all circumscribed polygons, is necessarily true of the circle.

478. Theorem.—*A curve is shorter than any other line which joins its ends, and toward which it is convex.*

For the curve BDC is the limit of those broken lines which have their vertices in it. Then, the curve BDC is less than the line BFC (79).



479. Corollary.—The circumference of a circle is shorter than the perimeter of a circumscribed polygon.

480. Corollary.—The circumference of a circle is longer than the perimeter of an inscribed polygon.

This is a corollary of the Axiom of Distance (54).

481. Theorem.—*A circle has a less perimeter than any equivalent polygon.*

For, of equivalent polygons, that has the least perimeter which is regular (468), and has the greatest number of sides (470).

482. Corollary.—A circle has a greater area than any isoperimetrical figure.

CIRCLES SIMILAR.

483. Theorem.—*Circles are similar figures.*

For angles which intercept like parts of a circumference are equal (197 and 224). Hence, whatever lines be made in one circle, homologous lines, making equal angles, may be made in another.

This theorem may be otherwise demonstrated, thus: Inscribed regular polygons of the same number of sides are similar. The number of sides may be increased indefinitely, and the polygons will still be similar at each successive step. The circles being the limits of the polygons, must also be similar.

484. Theorem.—*Two sectors are similar when the angles made by their radii are equal.*

485. Theorem.—*Two segments are similar when the angles which are formed by radii from the ends of their respective arcs are equal.*

These two theorems are demonstrated by completing the circles of which the given figures form parts. Then the given straight lines in one circle are homologous to those in the other; and any angle in one may have its corresponding equal angle in the other, since the circles are similar.

EXERCISE.

486. When the Tyrian Princess stretched the thongs cut from the hide of a bull around the site of Carthage, what course should she have pursued in order to include the greatest extent of territory?

RECTIFICATION OF CIRCUMFERENCE.

487. Theorem.—*The ratio of the circumference to its diameter is a constant quantity.*

Two circumferences are to each other in the ratio of their diameters. For the perimeters of similar regular polygons are in the ratio of homologous lines (435); and the circumference is the limit of the perimeters of regular polygons (476). Then, designating any two circumferences by C and C′, and their diameters by D and D′,

$$C : C' :: D : D'.$$

Hence, by alternation,

$$C : D :: C' : D'.$$

That is, the ratio of a circumference to its diameter is the same as that of any other circumference to its diameter.

488. The ratio of the circumference to the diameter is usually designated by the Greek letter π, the initial of perimeter.

If we can determine this numerical ratio, multiplying any diameter by it will give the circumference, or a straight line of the same extent as the circumference. This is called the *rectification* of that curve.

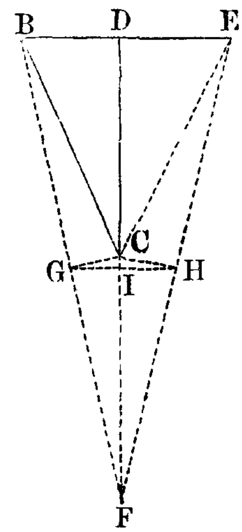
489. The number π is less than 4 and greater than 3. For, if the diameter is 1, the perimeter of the circumscribed square is 4; but this is greater than the circumference (479). And the perimeter of the inscribed regular hexagon is 3, but this is less than the circumference (480).

In order to calculate this number more accurately, let us first establish these two principles:

490. Theorem.—*Given the apothem, radius, and side of a regular polygon; the apothem of a regular polygon of the same length of perimeter, but double the number of sides, is half the sum of the given apothem and radius; and the radius of the polygon of double the number of sides, is a mean proportional between its own apothem and the given radius.*

Let CD be the apothem, CB the radius, and BE the side of a regular polygon. Produce DC to F, making

CF equal to CB. Join BF and EF. From C let the perpendicular CG fall upon BF. Make GH parallel to BE, and join CH and CE.



Now, the triangle BCF being isosceles by construction, the angles CBF and CFB are equal. The sum of these two is equal to the exterior angle BCD (261). Hence, the angle BFD is half the angle BCD. Since DF is, by hypothesis, perpendicular to BE at its center, BCE and BFE are isosceles triangles (108), and the angles BCE and BFE are bisected by the line DF (271). Therefore, the angle BFE is half the angle BCE. That is, the angle BFE is equal to the angle at the center of a regular polygon of double the number of sides of the given polygon (447).

Since GH is parallel to BE,

We have, GH : BE : : GF : BF.

Since GF is the half of BF (271), GH is the half of BE. Then GH is equal to the side of a regular polygon, with the same length of perimeter as the given polygon, and double the number of sides.

Again, FH and FG, being halves of equals, are equal. Also, IF is perpendicular to GH (127). Therefore, we have GH the side, IF the apothem, and GF the radius of the polygon of double the number of sides, with a perimeter equal to that of the given polygon.

Now, the similar triangles give,

FI : FD : : FG : FB.

Therefore, FI is one-half of FD. But FD is, by construction, equal to the sum of CD and CB. Therefore, the apothem of the second polygon is equal to half the sum of the given apothem and radius.

Again, in the right angled triangle GCF (324),

$$FC : FG :: FG : FI.$$

But FC is equal to CB; therefore, FG, the radius of the second polygon, is a mean proportional between the given radius and the apothem of the second.

491. For convenient application of these principles, let us represent the given apothem by a, the radius by r, and the side by s, the apothem of the polygon of double the number of sides by x, and its radius by y.

Then, $x=\frac{a+r}{2}$, and $x : y :: y : r$.

Hence, $y^2=xr$, and $y=\sqrt{xr}$.

492. Again, since, in any regular polygon, the apothem, radius, and half the side form a right angled triangle,

We always have, $r^2=a^2+\left(\frac{s}{2}\right)^2$

Hence, $a=\sqrt{r^2-\frac{s^2}{4}}=\frac{1}{2}\sqrt{4r^2-s^2}.$

493. Problem.—*To find the approximate value of the ratio of the circumference to the diameter of a circle.*

Suppose a regular hexagon whose perimeter is unity. Then its side is $\frac{1}{6}$ or .166667, and its radius is the same (454).

By the formula, $a=\frac{1}{2}\sqrt{4r^2-s^2}$, the apothem is

$$\frac{1}{2}\sqrt{\tfrac{4}{36}-\tfrac{1}{36}}=\tfrac{1}{12}\sqrt{3}, \text{ or } .144338.$$

Then, by the formula, $x=\frac{1}{2}(a+r)$, the apothem of the regular polygon of twelve sides, the perimeter being unity, is $\frac{1}{2}\left(\frac{1}{6}+\frac{1}{12}\sqrt{3}\right)$ or .155502. The radius of the same, by the formula $y = \sqrt{xr}$, is .160988. Proceeding in the same way, the following table may be constructed:

REGULAR POLYGONS WHOSE PERIMETER IS UNITY.

Number of sides.	Apothem.	Radius.
6	.144338	.166667
12	.155502	.160988
24	.158245	.159610
48	.158928	.159269
96	.159098	.159183
192	.159141	.159162
384	.159151	.159157
768	.159154	.159155
1536	.159155	.159155

Now, observe that the numbers in the second column express the ratios of the radius of any circle to the perimeters of the circumscribed regular polygons; and that those in the third column express the ratios of the radius to the perimeters of the inscribed polygons. These ratios gradually approach each other, till they agree for six places of decimals. It is evident that by continuing the table, and calculating the ratios to a greater number of decimal places, this approximation could be made as near as we choose.

But it has been already shown that the circumference is less than the perimeter of the circumscribed, and greater than that of the inscribed polygon. Hence, we conclude, that when the circumference is 1, the radius is .159155, with a near approximation to exactness. The diameter, being double the radius, is .31831.

Therefore,

$$\pi = \frac{1}{.31831} = 3.14159.$$

494. It was shown by *Archimedes*, by methods resembling the above, that the value of π is less than $3\frac{1}{7}$, and greater than $3\frac{10}{71}$. This number, $3\frac{1}{7}$, is in very common use for mechanical purposes. It is too great by about one eight-hundredth of the diameter.

About the year 1640, *Adrian Metius* found the nearer approximation $\frac{355}{113}$, which is true for six places of decimals. It is easily retained in the memory, as it is composed of the first three odd numbers, in pairs, 113,355, taking the first three digits for the denominator, and the other three for the numerator.

By the integral calculus, it has been found that π is equal to the series $4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} +$, etc.

By the calculus also, other and shorter methods have been discovered for finding the approximate value of π. In 1853, *Mr. Rutherford* presented to the Royal Society of London a calculation of the value of π to five hundred and thirty decimals, made by *Mr. W. Shanks*, of Houghton-le-Spring.

The first thirty-nine decimals are,

3.141 592 653 589 793 238 462 643 383 279 502 884 197.

EXERCISES.

495.—1. Two wheels, whose diameters are twelve and eighteen inches, are connected by a belt, so that the rotation of one causes that of the other. The smaller makes twenty-four rotations in a minute; what is the velocity of the larger wheel?

2. Two wheels, whose diameters are twelve and eighteen inches, are fixed on the same axle, so that they turn together. A point on the rim of the smaller moves at the rate of six feet per second; what is the velocity of a point on the rim of the larger wheel?

3. If the radius of a car-wheel is thirteen inches, how many revolutions does it make in traveling one mile?

4. If the equatorial diameter of the earth is 7924 miles, what is the length of one degree of longitude on the equator?

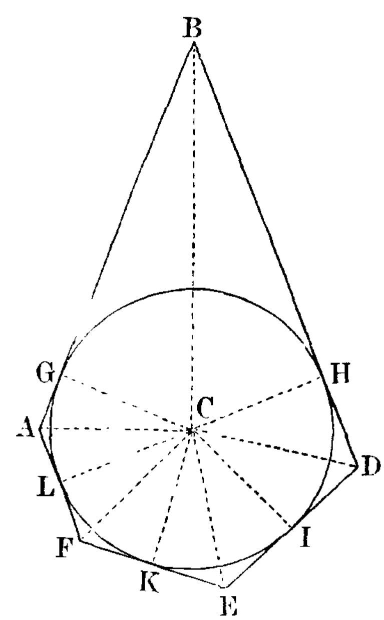
QUADRATURE OF CIRCLE.

496. The *quadrature* or *squaring of the circle*, that is, the finding an equivalent rectilinear figure, is a problem which excited the attention of mathematicians during many ages, until it was demonstrated that it could only be solved approximately.

The solution depends, indeed, on the rectification of the circumference, and upon the following

497. Theorem.—*The area of any polygon in which a circle can be inscribed, is measured by half the product of its perimeter by the radius of the inscribed circle.*

From the center C of the circle, let straight lines extend to all the vertices of the polygon ABDEF, also to all the points of tangency, G, H, I, K, and L.



The lines extending to the points of tangency are radii of the circle, and are therefore perpendicular to the sides of the polygon, which are tangents of the circle (183). The polygon is divided by the lines extending to the vertices into as many triangles as it has sides, ACB, BCD, etc. Regarding the sides of the polygon, AB, BD, etc., as the bases of these several triangles, they all have equal altitudes, for the radii are perpendicular to the sides of the polygon. Now, the area of each triangle is measured by half the product of its base by the common altitude. But the area of the polygon is the sum of the areas of the triangles, and the perimeter of the polygon is the sum of their bases. It follows that the area of the polygon is measured by half the product of the perimeter by the common altitude, which is the radius.

498. Corollary.—The area of a regular polygon is measured by half the product of its perimeter by its apothem.

499. Theorem.—*The area of a circle is measured by half the product of its circumference by its radius.*

For the circle is the limit of all the polygons that may be circumscribed about it, and its circumference is the limit of their perimeters.

500. Theorem.—*The area of a circle is equal to the square of its radius, multiplied by the ratio of the circumference to the diameter.*

For, let r represent the radius. Then, the diameter is $2r$, and the circumference is $\pi \times 2r$, and the area is $\frac{1}{2}\pi \times 2r \times r$, or πr^2 (499); that is, the square of the radius multiplied by the ratio of the circumference to the diameter.

501. Corollary.—The areas of two circles are to each other as the squares of their radii; or, as the squares of their diameters.

502. Corollary.—When the radius is unity, the area is expressed by π.

503. Theorem.—*The area of a sector is measured by half the product of its arc by its radius.*

For, the sector is to the circle as its arc is to the circumference. This may be proved in the same manner as the proportionality of arcs and angles at the center (197 or 202).

504. Since that which is true of every polygon may be shown, by the method of limits, to be true also of plane figures bounded by curves, it follows that in any two similar plane surfaces the ratio of the areas is the second power of the linear ratio.

505. Some of the following exercises are only arithmetical applications of geometrical principles.

The algebraic method may be used to great advantage in many exercises, but every principle or solution that is found in this way, should also be demonstrated by geometrical reasoning.

EXERCISES.

506.—1. What is the length of the radius when the arc of 80° is 10 feet?

2. What is the value, in degrees, of the angle at the center, whose arc has the same length as the radius?

3. What is the area of the segment, whose arc is 60°, and radius 1 foot?

4. To divide a circle into two or more equivalent parts by concentric circumferences.

5. One-tenth of a circular field, of one acre, is in a walk extending round the whole; required the width of the walk.

6. Two irregular garden-plats, of the same shape, contain, respectively, 18 and 32 square yards; required their linear ratio.

7. To describe a circle equivalent to two given circles.

507. The following exercises may require the student to review the leading principles of Plane Geometry.

1. From two points, one on each side of a given straight line, to draw lines making an angle that is bisected by the given line.

2. If two straight lines are not parallel, the difference between the alternate angles formed by any secant, is constant.

3. To draw the minimum tangent from a given straight line to a given circumference.

4. How many circles can be made tangent to three given straight lines?

5. Of all triangles on the same base, and having the same vertical angle, the isosceles has the greatest area.

6. To describe a circumference through a given point, and touching a given line at a given point.

7. To describe a circumference through two given points, and touching a given straight line.

8. To describe a circumference through a given point, and touching two given straight lines.

9. About a given circle to describe a triangle similar to a given triangle.

10. To draw lines having the ratios $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc.

11. To construct a triangle with angles in the ratio 1, 2, 3.

12. Can two unequal triangles have a side and two angles in the one equal to a side and two angles in the other?

13. To construct a triangle when the three lines extending from the vertices to the centers of the opposite sides are given?

14. If two circles touch each other, any two straight lines extending through the point of contact will be cut proportionally by the circumferences.

15 If any point on the circumference of a circle circumscribing an equilateral triangle, be joined by straight lines to the several vertices, the middle one of these lines is equivalent to the other two.

16. Making two diagonals in any quadrilateral, the triangles formed by one have their areas in the ratio of the parts of the other.

17. To bisect any quadrilateral by a line from a given vertex.

18. In the triangle ABC, the side $AB = 13$, $BC = 15$, the altitude $= 12$; required the base AC.

19. The sides of a triangle have the ratio of 65, 70, and 75; its area is 21 square inches; required the length of each side.

20. To inscribe a square in a given segment of a circle.

21. If any point within a parallelogram be joined to each of the four vertices, two opposite triangles, thus formed, are together equivalent to half the parallelogram.

22. To divide a straight line into two such parts that the rectangle contained by them shall be a maximum.

23. The area of a triangle which has one angle of 30°, is one-fourth the product of the two sides containing that angle.

24. To construct a right angled triangle when the area and hypotenuse are given.

25. Draw a right angle by means of Article 413.

26. To describe four equal circles, touching each other exteriorly, and all touching a given circumference interiorly.

27. A chord is 8 inches, and the altitude of its segment 3 inches; required the area of the circle.

28. What is the area of the segment whose arc is 36°, and chord 6 inches?

29. The lines which bisect the angles formed by producing the sides of an inscribed quadrilateral, are perpendicular to each other.

30. If a circle be described about any triangle ABC, then, taking BC as a base, the side AC is to the altitude of the triangle as the diameter of the circle is to the side AB.

31. By the proportion just stated, show that the area of a triangle is measured by the product of the three sides multiplied together, divided by four times the radius of the circumscribing circle.

32. In a quadrilateral inscribed in a circle, the sum of the two rectangles contained by opposite sides, is equivalent to the rectangle contained by the diagonals. This is known as the *Ptolemaic Theorem.*

33. Twice the square of the straight line which joins the vertex of a triangle to the center of the base, added to twice the square of half the base, is equivalent to the sum of the squares of the other two sides.

34. The sum of the squares of the sides of any quadrilateral is equivalent to the sum of the squares of the diagonals, increased by four times the square of the line joining the centers of the diagonals.

35. If, from any point in a circumference, perpendiculars be let fall on the sides of an inscribed triangle, the three points of intersection will be in the same straight line.

GEOMETRY OF SPACE.

CHAPTER IX.

STRAIGHT LINES AND PLANES.

508. The elementary principles of those geometrical figures which lie in one plane, furnish a basis for the investigation of the properties of those figures which do not lie altogether in one plane.

We will first examine those straight figures which do not inclose a space; after these, certain solids, or inclosed portions of space.

The student should bear in mind that when straight lines and planes are given by position merely, without mentioning their extent, it is understood that the extent is unlimited.

LINES IN SPACE.

509. Theorem.—*Through a given point in space there can be only one line parallel to a given straight line.*

This theorem depends upon Articles 49 and 117, and includes Article 119.

510. Theorem.—*Two straight lines in space parallel to a third, are parallel to each other.*

This is an immediate consequence of the definition of parallel lines, and includes Article 118.

511. Problem.—*There may be in space any number of straight lines, each perpendicular to a given straight line at one point of it.*

For we may suppose that while one of two perpendicular lines remains fixed as an axis, the other revolves around it, remaining all the while perpendicular (48). The second line can thus take any number of positions.

This does not conflict with Article 103, for, in this case, the axis is not in the same plane with any two of the perpendiculars.

EXERCISES.

512.—1. Designate two lines which are everywhere equally distant, but which are not parallel.

2. Designate two straight lines which are not parallel, and yet can not meet.

3. Designate four points which do not lie all in one plane.

PLANE AND LINES.

513. Theorem.—*The position of a plane is determined by any plane figure except a straight line.*

This is a corollary of Article 60.

Hence, we say, the plane of an angle, of a circumference, etc.

514. Theorem.—*A straight line and a plane can have only one common point, unless the line lies wholly in the plane.*

This is a corollary of Article 58.

515. When a line and a plane have only one common point, the line is said to *pierce* the plane, and the plane to *cut* the line. The common point is called the *foot* of the line in the plane.

When a line lies wholly in a plane, the plane is said to *pass through* the line.

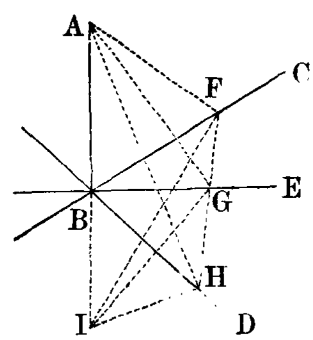
516. Theorem.—*The intersection of two planes is a straight line.*

For two planes can not have three points common, unless those points are all in one straight line (59).

PERPENDICULAR LINES.

517. Theorem.—*A straight line which is perpendicular to each of two straight lines at their point of intersection, is perpendicular to every other straight line which lies in the plane of the two, and passes through their point of intersection.*

In the diagram, suppose D, B, and C to be on the plane of the paper, the point A being above, and I below that plane.



If the line AB is perpendicular to BC and to BD, it is also perpendicular to every other line lying in the plane of DBC, and passing through the point B; as, for example, BE.

Produce AB, making BI equal to BA, and let any line, as FH, cut the lines BC, BE, and BD, in F, G, and H. Then join AF, AG, AH, and IF, IG, and IH.

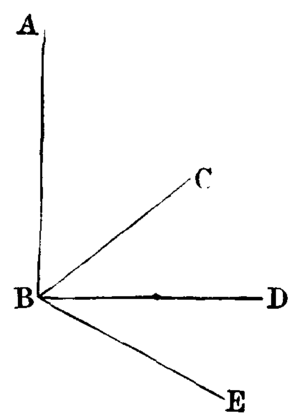
Now, since BC and BD are perpendicular to AI at its center, the triangles AFH and IFH have AF equal to IF (108), AH equal to IH, and FH common. Therefore, they are equal, and the angle AHF is equal to IHF. Then the triangles AHG and IHG are equal (284), and the lines AG and IG are equal. Therefore, the line BG, having two points each equally distant from A and I, is perpendicular to the line AI at its center B (109).

In the same way, prove that any other line through B, in the plane of DBC, is perpendicular to AB.

518. Theorem.—*Conversely, if several straight lines are each perpendicular to a given line at the same point, then these several lines all lie in one plane.*

Thus, if BA is perpendicular to BC, to BD, and to BE, then these three all lie in one plane.

BD, for instance, must be in the plane CBE. For the intersection of the plane of ABD with the plane of CBE is a straight line (516). This straight intersection is perpendicular to AB at the point B (517). Therefore, it coincides with BD (103). Thus it may be shown that any other line, perpendicular to AB at the point B, is in the plane of C, B, D, and E.



519. A straight line is said to be *perpendicular to a plane* when it is perpendicular to every straight line which passes through its foot in that plane, and the plane is said to be *perpendicular to the line.* Every line not perpendicular to a plane which cuts it, is called *oblique.*

520. Corollary.—If a plane cuts a line perpendicularly at the middle point of the line, then every point of the plane is equally distant from the two ends of the line (108).

521. Corollary.—If one of two perpendicular lines revolves about the other, the revolving line describes a plane which is perpendicular to the axis.

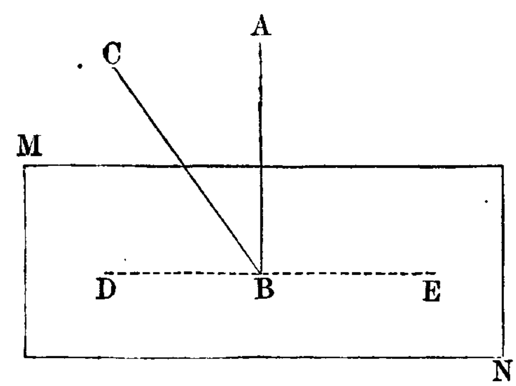
522. Corollary.—Through one point of a straight line there can be only one plane perpendicular to that line.

523. Theorem.—*Through a point out of a plane there can be only one straight line perpendicular to the plane.*

For, if there could be two perpendiculars, then each would be perpendicular to the line in the plane which joins their feet (519). But this is impossible (103).

524. Theorem.—*Through a point in a plane there can be only one straight line perpendicular to the plane.*

Let BA be perpendicular to the plane MN at the point B. Then any other line, BC for example, will be oblique to the plane MN.



For, if the plane of ABC be produced, its intersection with the plane MN will be a straight line.

Let DE be this intersection. Then AB is perpendicular to DE. Hence, BC, being in the plane of A, D, and E, is not perpendicular to DE (103). Therefore, it is not perpendicular to the plane MN (519).

525. Corollary.—The direction of a straight line in space is fixed by the fact that it is perpendicular to a given plane.

The directions of a plane are fixed by the fact that it is perpendicular to a given line.

526. Corollary.—All straight lines which are perpendicular to the same plane, have the same direction; that is, they are parallel to each other.

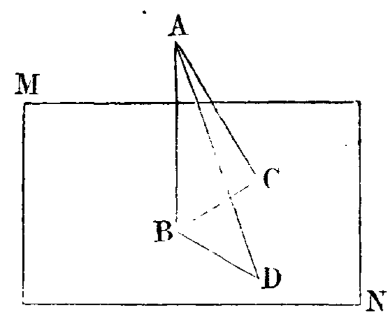
527. Corollary.—If one of two parallel lines is perpendicular to a plane, the other is also.

528. The AXIS of a circle is the straight line perpendicular to the plane of the circle at its center.

OBLIQUE LINES AND PLANES.

529. Theorem.—*If from a point without a plane, a perpendicular and oblique lines be extended to the plane, then two oblique lines which meet the plane at equal distances from the foot of the perpendicular, are equal.*

Let AB be perpendicular, and AC and AD oblique to the plane MN, and the distances BC and BD equal.



Then the triangles ABC and ABD are equal (284), and AC is equal to AD.

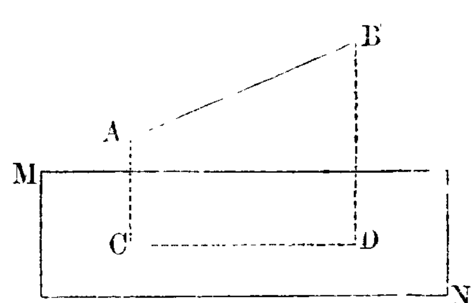
530. Corollary.—A perpendicular is the shortest line from a point to a plane. Hence, the distance from a point to a plane is measured by a perpendicular line.

531. Corollary.—All points of the circumference of a circle are equidistant from any point of its axis.

532. If from all points of a line perpendiculars be let fall upon a plane, the line thus described upon the plane is the *projection* of the given line upon the given plane.

533. Theorem.—*The projection of a straight line upon a plane is a straight line.*

Let AB be the given line, and MN the given plane. Then, from the points A and B, let the perpendiculars, AC and BD, fall upon the plane MN. Join CD.



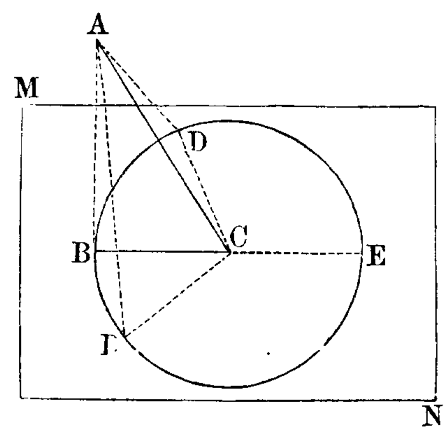
AC and BD, being perpendicular to the same plane, are parallel (526), and lie in one plane (121). Now, every perpendicular to MN let fall from a point of AB, must be parallel to BD, and must therefore lie in the plane AD, and meet the plane MN in some point of CD. Hence, the straight line CD is the projection of the straight line AB on the plane MN.

There is one exception to this proposition. When the given line is perpendicular to the plane, its projection is a point.

534. Corollary.—A straight line and its projection on a plane, both lie in one plane.

535. Theorem.—*The angle which a straight line makes with its projection on a plane, is smaller than the angle it makes with any other line in the plane.*

Let AC be the given line, and BC its projection on the plane MN. Then the angle ACB is less than the angle made by AC with any other line in the plane, as CD.



With C as a center and BC as a radius, describe a circumference in the plane MN, cutting CD at D.

Then the triangles ACD and ACB have two sides of the one respectively equal to two sides of the other. But the third side AD is longer than the third side AB (530). Therefore, the angle ACD is greater than the angle ACB (294).

536. Corollary.—The angle ACE, which a line makes with its projection produced, is larger than the angle made with any other line in the plane.

537. The angle which a line makes with its projection in a plane, is called the *Angle of Inclination* of the line and the plane.

PARALLEL LINES AND PLANE.

538. Theorem.—*If a straight line in a plane is parallel to a straight line not in the plane, then the second line and the plane can not have a common point.*

For if any line is parallel to a given line in a plane, and passes through any point of the plane, it will lie wholly in the plane (121). But, by hypothesis, the second line does not lie wholly in the plane. Therefore, it can not pass through any point of the plane, to whatever extent the two may be produced.

539. Such a line and plane, having the same direction, are called *parallel.*

540. Corollary.—If one of two parallel lines is parallel to a plane, the other is also.

541. Corollary.—A line which is parallel to a plane is parallel to its projection on that plane.

542. Corollary.—A line parallel to a plane is everywhere equally distant from it.

APPLICATIONS.

543. Three points, however placed, must always be in the same plane. It is on this principle that stability is more readily obtained by three supports than by a greater number. A three-legged stool *must* be steady, but if there be four legs, their ends should be in one plane, and the floor should be level. Many surveying and astronomical instruments are made with three legs.

544. The use of lines perpendicular to planes is very frequent in the mechanic arts. A ready way of constructing a line perpendicular to a plane is by the use of two squares (114). Place the angle of each at the foot of the desired perpendicular, one side of each square resting on the plane surface. Bring their perpendicular sides together. Their position must then be that of a perpendicular to the plane, for it is perpendicular to two lines in the plane.

545. When a circle revolves round its axis, the figure undergoes no real change of position, each point of the circumference taking successively the position deserted by another point.

On this principle is founded the operation of millstones. Two circular stones are placed so as to have the same axis, to which their faces are perpendicular, being, therefore, parallel to each other. The lower stone is fixed, while the upper one is made to revolve. The relative position of the faces of the stones undergoes no change during the revolution, and their distance being properly regulated, all the grain which passes between them will be ground with the same degree of fineness.

546. In the turning lathe, the axis round which the body to be turned is made to revolve, is the axis of the circles formed by the cutting tool, which removes the matter projecting beyond a proper distance from the axis. The cross section of every part of the thing turned is a circle, all the circles having the same axis.

DIEDRAL ANGLES.

547. A Diedral Angle is formed by two planes meeting at a common line. This figure is also called simply a *diedral.* The planes are its *faces*, and the intersection is its *edge.*

In naming a diedral, four letters are used, one in each face, and two on the edge, the letters on the edge being between the other two.

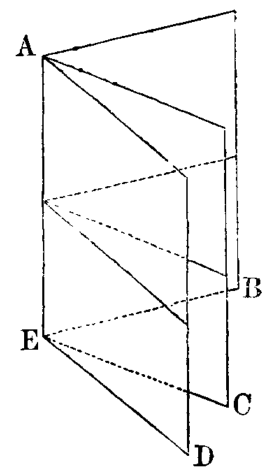
This figure is called a diedral *angle*, because it is similar in many respects to an angle formed by two lines.

MEASURE OF DIEDRALS.

548. The quantity of a diedral, as is the case with a linear angle, depends on the difference in the directions of the faces from the edge, without regard to the extent of the planes. Hence, two diedrals are equal when they can be so placed that their planes will coincide.

549. Problem.—*One diedral may be added to another.*

In the diagram, AB, AC, and AD represent three planes having the common intersection AE.



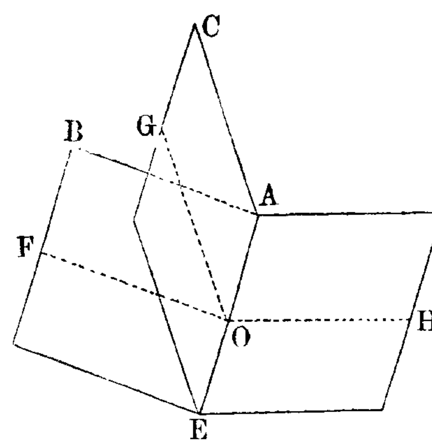
Evidently the sum of BEAC and CEAD is equal to BEAD.

550. Corollary. — Diedrals may be subtracted one from another. A diedral may be bisected or divided in any required ratio by a plane passing through its edge.

551. But there are in each of these planes any number of directions. Hence, it is necessary to determine which of these is properly the direction of the face from the edge. For this purpose, let us first establish the following principle:

552. Theorem.—*One diedral is to another as the plane angle, formed in the first by a line in each face perpendicular to the edge, is to the similarly formed angle in the other.*

Thus, if FO, GO, and HO are each perpendicular to AE, then the diedral CEAD is to the diedral BEAD as the angle GOH is to the angle FOH. This may be demonstrated in the same manner as the proposition in Article 197.



553. Corollary.—A diedral is said to be measured by the plane angle formed by a line in each of its faces perpendicular to the edge.

554. Corollary.—Accordingly, a diedral angle may be acute, obtuse, or right. In the last case, the planes are perpendicular to each other.

555. Many of the principles of plane angles may be applied to diedrals, without further demonstration.

All right diedral angles are equal (90).

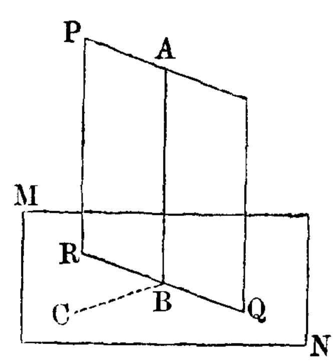
When the sum of several diedrals is measured by two right angles, the outer faces form one plane (100).

When two planes cut each other, the opposite or vertical diedrals are equal (99).

PERPENDICULAR PLANES.

556. Theorem.—*If a line is perpendicular to a plane, then any plane passing through this line is perpendicular to the other plane.*

If AB in the plane PQ is perpendicular to the plane MN, then AB must be perpendicular to every line in MN which passes through the point B (519); that is, to RQ, the intersection of the two planes, and to BC, which is made perpendicular to the intersection RQ. Then, the angle ABC measures the inclination of the two planes (553), and is a right angle. Therefore, the planes are perpendicular.



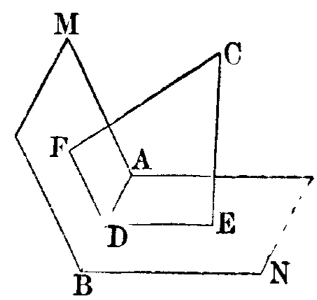
557. Corollary.—Conversely, if a plane is perpendicular to another, a straight line, which is perpendicular to one of them, at some point of their intersection, must lie wholly in the other plane (524).

558. Corollary.—If two planes are perpendicular to a third, then the intersection of the first two is a line perpendicular to the third plane.

OBLIQUE PLANES.

559. Theorem.—*If from a point within a diedral, perpendicular lines be made to the two faces, the angle of these lines is supplementary to the angle which measures the diedral.*

Let M and N be two planes whose intersection is AB, and CF and CE perpendiculars let fall upon them from the point C; and DF and DE the intersections of the plane FCE with the two planes M and N. Then the plane FCE must be perpendicular to each of the planes M and N (556).



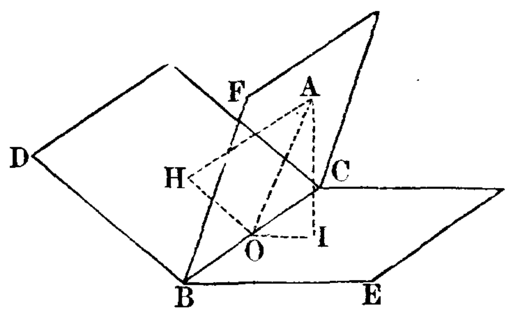
Hence, the line AB is perpendicular to the plane FCE (558), and the angles ADF and ADE are right angles. Then the angle FDE measures the diedral. But in the quadrilateral FDEC, the two angles F and E are right angles. Therefore, the other two angles at C and D are supplementary.

560. Theorem.—*Every point of a plane which bisects a diedral is equally distant from its two faces.*

Let the plane FC bisect the diedral DBCE. Then it is to be proved that every point of this plane, as A, for example, is equally distant from the planes DC and EC.

From A let the perpendiculars AH and AI fall upon the faces DC and EC, and let IO, AO, and HO be the intersections of the plane of the angle IAH with the three given planes.

Then it may be shown, as in the last theorem, that the angle HOA measures the diedral FBCD, and the angle IOA the diedral FBCE. But these diedrals are equal, by hypothesis. Therefore, the line AO bisects the angle IOH, and the point A is equally distant from the lines OH and OI (113). But the distance of A from these lines is measured by the same perpendiculars, AH and AI, which measure its distance from the two faces DC and EC. Therefore, any point of the bisecting plane is equally distant from the two faces of the given diedral.



APPLICATIONS.

561. Articles 548 to 554 are illustrated by a door turning on its hinges. In every position it is perpendicular to the floor and ceiling. As it turns, it changes its inclination to the wall, in which it is constructed, the angle of inclination being that which is formed by the upper edge of the door and the lintel.

562. The theory of diedrals is as important in the study of magnitudes bounded by planes, as is the theory of angles in the study of polygons.

This is most striking in the science of crystallography, which teaches us how to classify mineral substances according to their geometrical forms. Crystals of one kind have edges of which the diedral angles measure a certain number of degrees, and crystals of another kind have edges of a different number of degrees. Crystals of many species may be thus classified, by measuring their diedrals.

563. The plane of the surface of a liquid at rest is called *horizontal*, or the plane of the horizon. The direction of a plumbline when the weight is at rest, is a *vertical line*. The vertical line is perpendicular to the horizon, the positions of both being governed by the same causes. Every line in the plane of the horizon, or parallel to it, is called a horizontal line, and every plane passing through a vertical line is called a *vertical plane*. Every vertical plane is perpendicular to the horizon.

Horizontal and vertical planes are in most frequent use. Floors, ceilings, etc., are examples of the former, and walls of the latter. The methods of using the builder's level and plummet to determine the position of these, are among the simplest applications of geometrical principles.

Civil engineers have constantly to observe and calculate the position of horizontal and vertical planes, as all objects are referred to these. The astronomer and the navigator, at every step, refer to the horizon, or to a vertical plane.

EXERCISES.

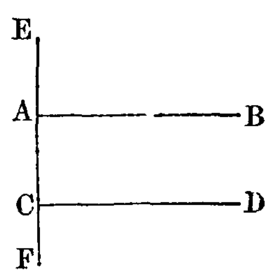
564.—1. If, from a point without a plane, several equal oblique lines extend to it, they make equal angles with the plane.

2. If a line is perpendicular to a plane, and if from its foot a perpendicular be let fall on some other line which lies in the plane, then this last line is perpendicular to the plane of the other two.

3. What is the locus of those points in space, each of which is equally distant from two given points?

PARALLEL PLANES.

565. Two planes which are perpendicular to the same straight line, at different points of it, are both fixed in position (525), and they have the same directions. If the parallel lines AB and CD revolve about the line EF, to which they are both perpendicular, then each of the revolving lines describes a plane.



Every direction assumed by one line is the same as that of the other, and, in the course of a complete revolution, they take all the possible directions of the two planes.

Two planes which have the same directions are called *parallel planes.*

PARALLELISM consists in having the same direction, whether it be of two lines, of two planes, or of a line and a plane.

566. Corollary.—Two planes parallel to a third are parallel to each other.

567. Corollary.—Two planes perpendicular to the same straight line are parallel to each other.

568. Corollary.—A straight line perpendicular to one of two parallel planes is perpendicular to the other.

569. Corollary.—Every straight line in one of two parallel planes has its parallel line in the other plane. Therefore, every straight line in one of the planes is parallel to the other plane.

570. Corollary.—Since through any point in a plane there may be a line parallel to any line in the same plane (121), therefore, in one of two parallel planes, and through any point of it, there may be a straight line parallel to any straight line in the other plane.

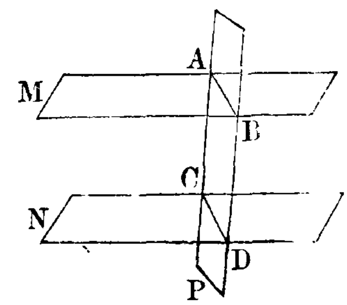
571. Theorem.—*Two parallel planes can not meet.*

For, if they had a common point, being parallel, they would have the same directions from that point, and therefore would coincide throughout, and be only one plane.

572. Theorem.—*The intersections of two parallel planes by a third plane are parallel lines.*

Let AB and CD be the intersections of the two parallel planes M and N, with the plane P.

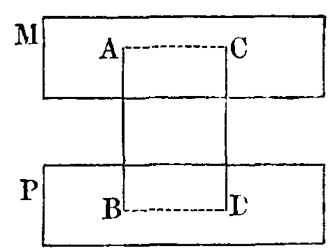
Now, if through C there be a line parallel to AB, it must lie in the plane P (121), and also in the plane N (570). Therefore, it is the intersection CD, and the two intersections are parallel lines.



When two parallel planes are cut by a third plane, eight diedrals are formed, which have properties similar to those of Articles 124 to 128.

573. Theorem.—*The parts of two parallel lines intercepted between parallel planes are equal.*

For, if the lines AB and CD are parallel, they lie in one plane. Then AC and BD are the intersections of this plane with the two parallel planes M and P. Hence, AC is parallel to BD, and AD is a parallelogram. Therefore, AB is equal to the opposite side CD.



574. Theorem.—*Two parallel planes are everywhere equally distant.*

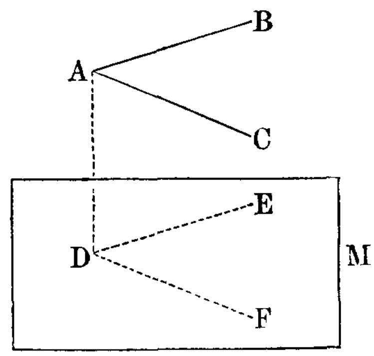
For the shortest distance from any point of one plane to the other, is measured by a perpendicular. But these perpendiculars are all parallel (526), and therefore equal to each other.

575. Theorem.—*If the two sides of an angle are each parallel to a given plane, then the plane of that angle is parallel to the given plane.*

If AB and AC are each parallel to the plane M, then the plane of BAC is parallel to the plane M.

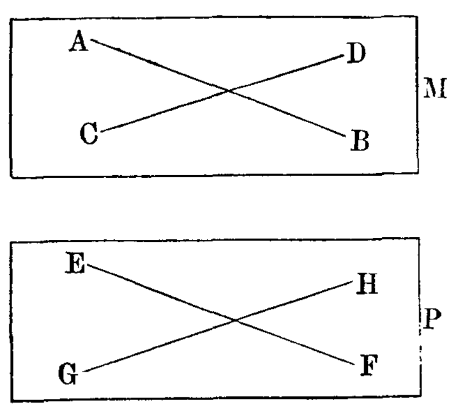
From A let the perpendicular AD fall upon the plane M, and let the projections of AB and AC on the plane M be respectively DE and DF.

Since DE is parallel to AB (541), DA is perpendicular to AB (127). For a like reason, DA is perpendicular to AC. Therefore, DA is perpendicular to the plane of BAC (517), and the two planes being perpendicular to the same line are parallel to each other (567).



576. Theorem.—*If two straight lines which cut each other are respectively parallel to two other straight lines which cut each other, then the plane of the first two is parallel to the plane of the second two.*

Let AB be parallel to EF, and CD parallel to GH. Then the planes M and P are parallel.



For AB being parallel to EF, is parallel to the plane P in which it lies (538). Also, CD is parallel to the plane P, for the same reason. Therefore, the plane M is parallel to the plane P (575).

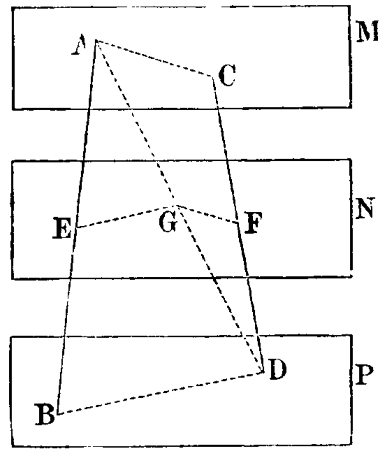
577. Corollary.—The angles made by the first two lines are respectively the same as those made by the second two. For they are the differences between the same directions.

This includes the corresponding principle of Plane Geometry.

578. Theorem.—*Straight lines cut by three parallel planes are divided proportionally.*

If the line AB is cut at the points A, E, and B, and the line CD at the points C, F, and D, by the parallel planes M, N, and P, then

AE : EB : : CF : FD.



Join AC, AD, and BD. AD pierces the plane N in the point G. Join EG and GF.

Now, EG and BD are parallel, being the intersections of the parallel planes N and P by the third plane ABD (572). Hence (313),

AE : EB : : AG : GD.

For a like reason,

AG : GD : : CF : FD.

Therefore, AE : EB : : CF : FD.

APPLICATION.

579. The general problem of *perspective* in drawing, consists in representing upon a plane surface the apparent form of objects in sight. This plane, the plane of the picture, is supposed to be between the eye and the objects to be drawn. Then each object is to be represented upon the plane, at the point where it would be pierced by the visual ray from the object to the eye.

All the visual rays from one straight object, such as the top of a wall, or one corner of a house, lie in one plane (60). Their intersection with the plane of the picture must be a straight line (516). Therefore, all straight objects, whatever their position, must be drawn as straight lines.

Two parallel straight objects, if they are also parallel to the plane of the picture, will remain parallel in the perspective. For the lines drawn must be parallel to the objects (572), and therefore to each other.

Two parallel lines, which are not parallel to the plane of the picture, will meet in the perspective. They will meet, if produced, at that point where the plane of the picture is pierced by a line from the eye parallel to the given lines.

EXERCISES.

580.—1. A straight line makes equal angles with two parallel planes.

2. Two parallel lines make the same angle of inclination with a given plane.

3. The projections of two parallel lines on a plane are parallel.

4. When two planes are each perpendicular to a third, and their intersections with the third plane are parallel lines, then the two planes are parallel to each other.

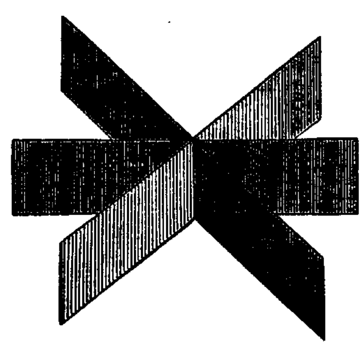
5. If two straight lines be not in the same plane, one straight line, and only one, may be perpendicular to both of them.

6. Demonstrate the last sentence of Article 579.

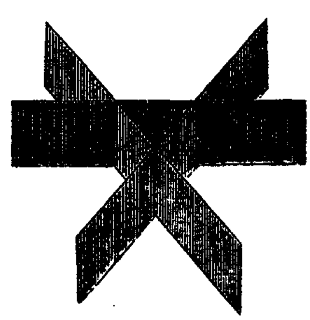
TRIEDRALS.

581. When three planes cut each other, three cases are possible.

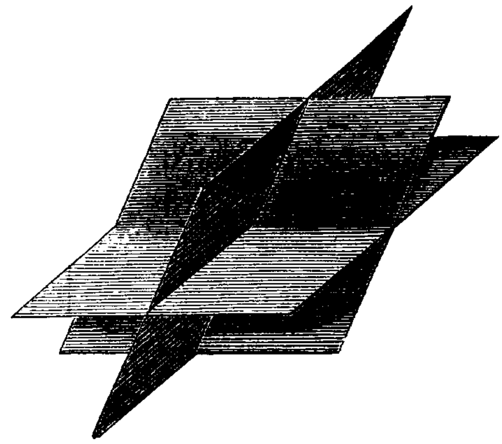
1st. The intersections may coincide. Then six diedrals are formed, having for their common edge the intersection of the three planes.



2d. The three intersections may be parallel lines. Then one plane is parallel to the intersection of the other two.



3d. The three intersections may meet at one point. Then the space about the point is divided by the three planes into eight parts.



The student will apprehend this better when he reflects that two intersecting planes make four diedrals. Now, if a third plane cut through the intersection of the first two, it will divide each of the diedrals into two parts, making eight in all. Each of these parts is called a triedral, because it has three faces.

A fourth case is impossible. For, since any two of the intersections lie in one plane, they must either be parallel, or they meet. If two of the intersections meet, the point of meeting must be common to the three planes, and must therefore be common to all the intersections. Hence, the three intersections either have more than one point common, only one point common, or no point common. But these are the three cases just considered.

582. A Triedral is the figure formed by three planes meeting at one point. The point where the planes and intersections all meet, is called the *vertex* of the triedral. The intersections are its *edges*, and the planes are its *faces*.

The corners of a room, or of a chest, are illustrations of triedrals with rectangular faces. The point of a triangular file, or of a small-sword, has the form of a triedral with acute faces.

The triedral has many things analogous to the plane triangle. It has been called a solid triangle; and more frequently, but with less propriety, a solid angle. The three faces, combined two and two, make three diedrals, and the three intersections, combined two and two, make three plane angles. These six are the six elements or principal parts of a triedral.

Each face is the plane of one of the plane angles, and two faces are said to be equal when these angles are equal.

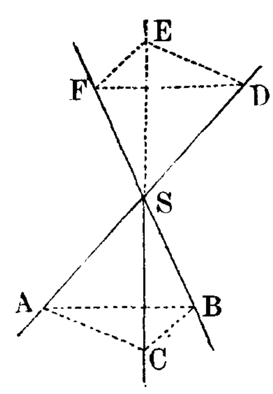
Two triedrals are said to be equal when their several planes may coincide, without regard to the extent of the planes. Since each plane is determined by two lines, it is evident that two triedrals are equal when their several edges respectively coincide.

583. A triedral which has one rectangular diedral, that is, whose measure is a right angle, is called a *rectangular triedral.* If it has two, it is *birectangular;* if it has three, it is *trirectangular.*

A triedral which has two of its faces equal, is called *isosceles;* if all three are equal, it is *equilateral.*

SYMMETRICAL TRIEDRALS.

584. If the edges of a triedral be produced beyond the vertex, they form the edges of a new triedral. The faces of these two triedrals are respectively equal, for the angles are vertical.



Thus, the angles ASC and ESD are equal; also, the angles BSC and FSE are equal, and the angles ASB and DSF.

The diedrals whose edges are FS and BS are also equal, since, being formed by the same planes, EFSBC and DFSBA, they are vertically opposite diedrals (555). The same is true of the diedrals whose edges are DS and SA, and of the diedrals whose edges are ES and SC.

In the diagram, suppose ASB to be the plane of the paper, C being above and E below that plane.

But the two triedrals are not equal, for they can not be made to coincide, although composed of parts which are respectively equal. This will be more evident if the student will imagine himself within the first triedral, his head toward the vertex, and his back to the plane ASB. Then the plane ASC will be on the right hand, and BSC on the left. Then let him imagine himself in the other triedral, his head toward the vertex, and his back to the plane FSD, which is equal to ASB. Then the plane on the right will be FSE, which is equal to BSC, the one that had been on the left; and the plane now on the left will be DSE, equal to the one that had been on the right.

Now, since the equal parts are not similarly situated, the two figures can not coincide.

Then the difference between these two triedrals consists in the opposite order in which the parts are arranged. This may be illustrated by two gloves, which we may suppose to be composed of exactly equal parts. But they are arranged in reverse order. The right hand glove will not fit the left hand. The two hands themselves are examples of the same kind.

585. When two magnitudes are composed of parts respectively equal, but arranged in reverse order, they are said to be *symmetrical magnitudes*.

The word symmetrical, as here used, has essentially the same meaning as that given in Plane Geometry (158). Two symmetrical plane figures, or parts of a figure, are divided by a straight line, while two such figures in space are divided by a plane.

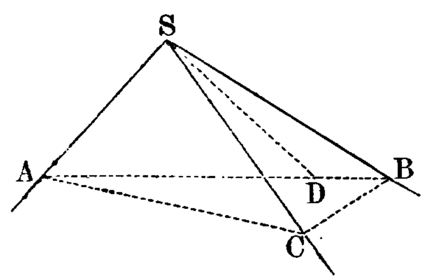
When two plane figures are symmetrical, they are also equal, for one can be turned over to coincide with the other, as with the figures *m* and *n* in Article 282. But this is not possible, as just shown, with figures that are not in one plane.

ANGLES OF A TRIEDRAL.

586. Theorem.—*Each plane angle of a triedral is less than the sum of the other two.*

The theorem is demonstrated, when it is shown that the greatest angle is less than the sum of the other two.

Let ASB be the largest of the three angles of the triedral S. Then, from the angle ASB take the part ASD, equal to the angle ASC. Join the edges SA and SB by any straight line AB. Take SC equal to SD, and join AC and BC.

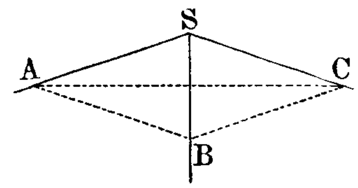


Since the triangles ASD and ASC are equal (284), AD is equal to AC. But AB is less than the sum of AC and BC, and from these, subtracting the equals AD and AC, we have BD less than BC. Hence, the triangles BSD and BSC have two sides of the one equal to two sides of the other, and the third side BD less than the third side BC. Therefore, the included angle BSD is less than the angle BSC. Adding to these the equal angles ASD and ASC, we have the angle ASB less than the sum of the angles ASC and BSC.

587. Theorem.—*The sum of the plane angles which form a triedral is always less than four right angles.*

Through any three points, one in each edge of the triedral, let the plane ABC pass, making the intersections AB, BC, and AC, with the faces.

There is thus formed a triedral at each of the points A, B, and C. Then the angle BAC is less than the sum of BAS and CAS (586). The angle ABC is less than the sum of ABS and CBS. The angle BCA is less than the sum of ACS and BCS. Adding together these inequalities, we find that the sum of the angles of the triangle ABC, which is two right angles, is less than the sum of the six angles at the bases of the triangles on the faces of the triedral S.



The sum of all the angles of these three triangles is six right angles. Therefore, since the sum of those at the bases is more than two right angles, the sum of those at the vertex S must be less than four right angles.

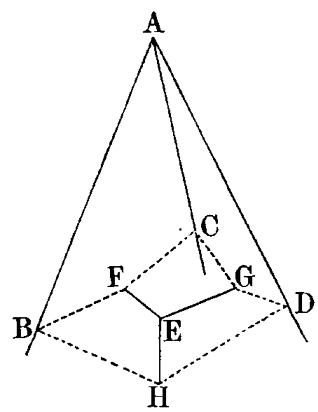
588. To assist the student to understand this theorem, let him take any three points on the paper or blackboard for A, B, and C. Take S at some distance from the surface, so that the plane angles formed at S will be quite acute. Then let S approach the surface of the triangle ABC. Evidently the angles at S become larger and larger, until the point S touches the surface of the triangle, when the sum of the angles becomes four right angles, and, at the same time, the triedral becomes one plane.

SUPPLEMENTARY TRIEDRALS.

589. Theorem.—*If, from a point within a triedral, perpendicular lines fall on the several faces, these lines* *will be the edges of a second triedral, whose faces will be supplements respectively of the diedrals of the first; and the faces of the first will be respectively supplements of the diedrals of the second triedral.*

A plane angle is not strictly the supplement of a diedral, but we understand, by this abridged expression, that the plane angle is the supplement of that which measures the diedral.

If from the point E, within the triedral ABCD, the perpendiculars EF, EG, and EH fall on the several faces, then these lines form a second triedral, whose faces are FEH, FEG, and GEH.



Then the angle FEH is the supplement of the diedral whose edge is BA, for the sides of the angle are perpendicular to the faces of the diedral (559). For the same reason, the angle FEG is the supplement of the diedral whose edge is CA, and the angle GEH is the supplement of the diedral whose edge is DA.

But it may be shown that these two triedrals have a reciprocal relation; that is, that the property just proved of the second toward the first, is also true of the first toward the second.

Let BF and BH be the intersections of the face FEH with the faces BAC and BAD; CF and CG be the intersections of the face FEG with the faces BAC and CAD; and DG and DH be the intersections of the face GEH with the faces CAD and BAD.

Now, since the plane FBH is perpendicular to each of the planes BAC and BAD (556), their intersection AB is perpendicular to the plane FBH (558). For a like reason, AC is perpendicular to the plane FCG and AD is perpendicular to the plane GDH. Then, reasoning as above, we prove that the angle BAC is the supplement of the diedral whose edge is FE; and that each of the other faces of the first triedral is a supplement of a diedral of the second.

590. Two triedrals, in which the faces and diedral angles of the one are respectively the supplements of the diedral angles and faces of the other, are called *supplementary triedrals.*

Instead of placing supplementary triedrals each within the other, as above, they may be supposed to have their vertices at the same point. Thus, at the point A, erect a perpendicular to each of the three faces of the triedral ABCD, and on the side of the face toward the triedral. A second triedral is thus formed, which is supplementary to the triedral ABCD, and is symmetrical to the one formed within.

SUM OF THE DIEDRALS.

591. Theorem.—*In every triedral the sum of the three diedral angles is greater than two right angles, and less than six.*

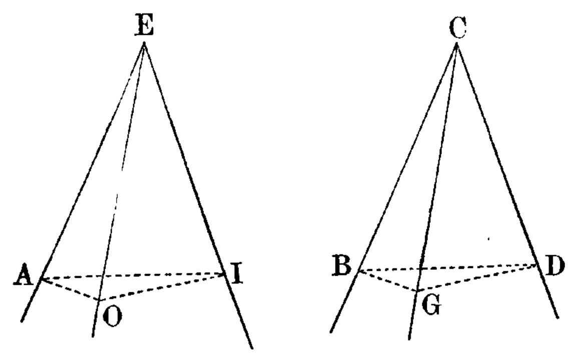
Consider the supplementary triedral, with the given one. Now, the sum of the three diedrals of the given triedral, and of the three faces of its supplementary triedral, must be six right angles; for the sum of each pair is two right angles. But the sum of the faces of the supplementary triedral is less than four right angles (587), and is greater than zero. Subtracting this sum from the former, the remainder, being the sum of the three diedrals of the given triedral, is greater than two and less than six right angles.

EQUALITY OF TRIEDRALS.

592. Theorem.—*When two triedrals have two faces, and the included diedral of the one respectively equal to the corresponding parts of the other, then the remaining face and diedrals of the first are respectively equal to the corresponding parts of the other.*

There are two cases to be considered.

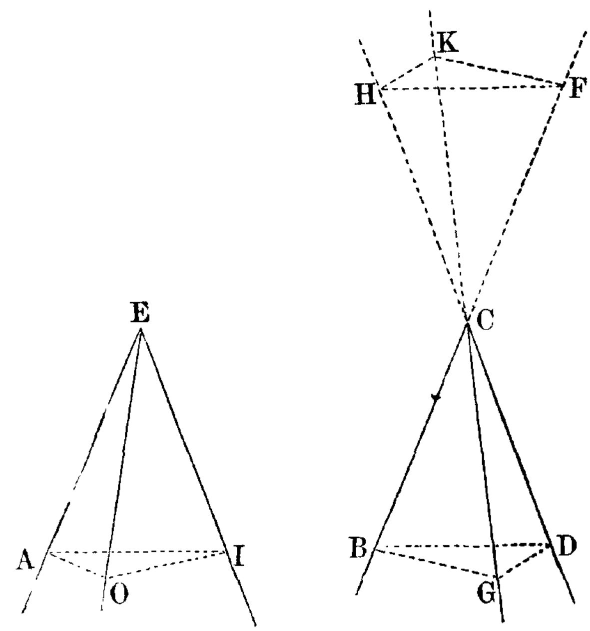
1st. Suppose the angles AEO and BCG equal, and the angles AEI and BCD equal, also the included diedrals whose edges are AE and BC. Let the arrangement be the same in both, so that, if we go around one triedral in the order O, A, I, O, and around the other in the order G, B, D, G, in both cases the triedral will be on the right. Then it may be proved that the two triedrals are equal.



Place the angle BCD directly upon its equal, AEI. Since the diedrals are equal, and are on the same side of the plane AEI, the planes BCG and AEO will coincide. Since the angles BCG and AEO are equal, the lines CG and EO will coincide. Thus, the angles DCG and IEO coincide, and the two triedrals coincide throughout.

2d. Let the angles AEO and DCG be equal, and the angles AEI and BCD, also the included diedrals, whose edges are AE and DC. But let the arrangement be reverse; that is, if we go around one triedral in the order O, A, I, O, and around the other in the order G, D, B, G, in one case the triedral will be to the right, and in the other it will be to the left of us. Then it may be proved that the two triedrals are symmetrical.

One of the triedrals can be made to coincide with the symmetrical of the other; for if the edges BC, GC, and DC be produced beyond C, the triedral CFHK will have two faces and the included diedral respectively equal to those parts of the triedral EAOI, and arranged in the same order; that is, the reverse of the triedral CDGB. Hence, as just shown, the triedrals CFHK and EAOI are equal.



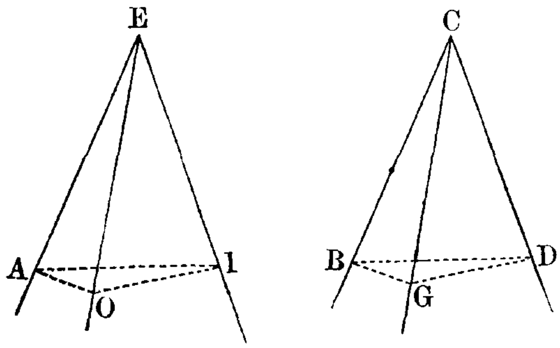
Therefore, EAOI and CDGB are symmetrical triedrals.

In both cases, all the parts of each triedral are respectively equal to those of the other.

593. Theorem.—*When two triedrals have one face and the two adjacent diedrals of the one respectively equal to the corresponding parts of the other, then the remaining faces and diedral of the first are respectively equal to the corresponding parts of the other.*

Suppose that the faces AEI and BCD are equal, that the diedrals whose edges are AE and BC are equal, that the diedrals whose edges are IE and DC are equal, and that these parts are similarly arranged in the two triedrals. Then the one may coincide with the other.

For BCD may coincide with its equal AEI, BC falling on AE. Then the plane of BCG must coincide with that of AEO, since the diedrals are equal; and the line CG will fall in the plane of AEO. For a similar reason CG will fall on the plane of IEO. Therefore, it must coincide with their intersection EO, and the two triedrals coincide throughout.

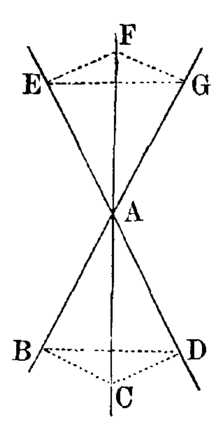


When the equal parts are in reverse order in the two triedrals, the arrangement in one must be the same as in the symmetrical of the other. Therefore, in that case, the two triedrals would be symmetrical.

In both cases, all the parts of each triedral are respectively equal to those of the other.

594. Theorem.—*An isosceles triedral and its symmetrical are equal.*

Let ABCD be an isosceles triedral, having the faces BAC and DAC equal, and let AEFG be its symmetrical triedral.



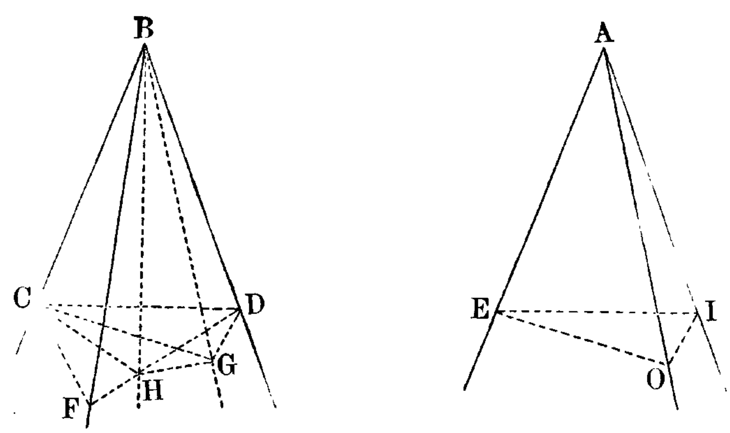
Now, the faces BAC, DAC, FAG, and FAE, are all equal to each other. The diedrals whose edges are AC and AF being vertical, are also equal. Hence, the faces mentioned being all equal, corresponding equal parts may be taken in the same order in both triedrals; that is, the face EAF equal to the face BAC, and the face FAG equal to CAD. Therefore, the two triedrals are equal.

595. Corollary.—In an isosceles triedral, the diedrals opposite the equal faces are equal. For the diedrals whose edges are AB and AD, are each equal to the diedral whose edge is AE.

596. Corollary.—Conversely, if in any triedral two of the diedral angles are equal, then the faces opposite these diedrals are equal, and the triedral is isosceles. For, as in the above theorem, the given triedral can be shown to be equal to its symmetrical.

597. Theorem.—*When two triedrals have two faces of the one respectively equal to two faces of the other, and the included diedrals unequal, then the third faces are unequal, and that face is greater which is opposite the greater diedral.*

Suppose that the faces CBD and EAI are equal, and



that the faces CBF and EAO are also equal, but that the diedral whose edge is CB is greater than the diedral whose edge is EA. Then the face FBD will be greater than the face OAI.

Through the line BC, let a plane GBC pass, making with the plane DBC a diedral equal to that whose edge is AE. In this plane, make the angle CBG equal to EAO. Let the diedral FBCG be bisected by the plane

HBC, BH being the intersection of this plane with the plane FBD.

Then the two triedrals BCDG and AEIO, having two faces and the included diedral in the one equal to the corresponding parts in the other, will have the remaining parts equal. Hence, the faces DBG and IAO are equal.

Again, the two triedrals BCFH and BCGH have the faces CBF and CBG equal, by construction, the face CBH common, and the included diedrals equal, by construction. Therefore, the third faces FBH and GBH are equal.

To each of these equals add the face HBD, and we have the face FBD equal to the sum of GBH and HBD. But in the triedral BDGH, the face DBG is less than the sum of the other two faces, GBH and HBD (586). Hence, the face DBG is less than FBD. Therefore, the face OAI, equal to DBG, is less than FBD.

598. Corollary.—Conversely, when two triedrals have two faces of the one respectively equal to two faces of the other, and the third faces are unequal, then the diedral opposite the greater face is greater than the diedral opposite the less.

599. Theorem.—*When two triedrals have their three faces respectively equal, their diedrals will be respectively equal; and the two triedrals are either equal, or they are symmetrical.*

When two faces of one triedral are respectively equal to those of another, if the included diedrals are unequal, then the opposite faces are unequal (597). But, by the hypothesis of this theorem, the third faces are equal. Therefore, the diedrals opposite to those faces must be equal.

In the same manner, it may be shown that the other diedral angles of the one, are equal to the corresponding diedral angles of the other triedral. Therefore, the triedrals are either equal or symmetrical, according to the arrangement of their parts.

600. Theorem. — *Two triedrals which have their diedrals respectively equal, have also their faces respectively equal; and the two triedrals are either equal, or they are symmetrical.*

Consider the supplementary triedrals of the two given triedrals. These will have their faces respectively equal, because they are the supplements of equal diedral angles (589). Since their faces are equal, their diedrals are equal (599). Then the two given triedrals, having their faces the supplements of these equal diedrals, will have those faces equal; and the triedrals are either equal or symmetrical, according to the arrangement of their parts.

601. The student may notice, in every other case of equal triedrals, the analogy to a case of equality of triangles; but the theorem just demonstrated has nothing analogous in plane geometry.

602. Corollary. — All trirectangular triedrals are equal.

603. Corollary.—In all cases where two triedrals are either equal or supplementary, equal faces are opposite equal diedral angles.

EXERCISES.

604.—1. In any triedral, the greater of two faces is opposite to the greater diedral angle; and conversely.

2. Demonstrate the principles stated in the last sentence of Article 590.

3. If a triedral have one right diedral angle, then an adjacent

face and its opposite diedral are either both acute, both right, or both obtuse.

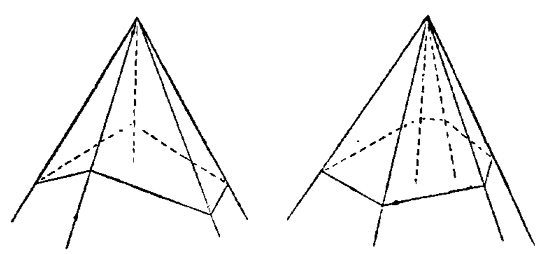
POLYEDRALS.

605. A POLYEDRAL is the figure formed by several planes which meet at one point. Thus, a polyedral is composed of several angles having their vertices at a common point, every edge being a side of two of the angular faces. The triedral is a polyedral of three faces.

606. Problem.—*Any polyedral of more than three faces may be divided into triedrals*

For a plane may pass through any two edges which are not adjacent. Thus, a polyedral of four faces may be divided into two triedrals; one of five faces, into three; and so on.

607. This is like the division of a polygon into triangles. The plane passing through two edges not adjacent is called a *diagonal plane.*



A polyedral is called *convex*, when every possible diagonal plane lies within the figure; otherwise it is called *concave.*

608. Corollary.—If the plane of one face of a convex polyedral be produced, it can not cut the polyedral.

609. Corollary.—A plane may pass through the vertex of a convex polyedral, without cutting any face of the polyedral.

610. Corollary.—A plane may cut all the edges of a convex polyedral. The section is a convex polygon.

611. When any figure is cut by a plane, the figure that is defined on the plane by the limits of the figure so cut, is called a *plane section.*

Several properties of triedrals are common to other polyedrals.

612. Theorem.—*The sum of all the angles of a convex polyedral is less than four right angles.*

For, suppose the polyedral to be cut by a plane, then the section is a polygon of as many sides as the polyedral has faces. Let n represent the number of sides of the polygon. The plane cuts off a triangle on each face of the polyedral, making n triangles. Now, the sum of the angles of this polygon is $2n - 4$ right angles (424), and the sum of the angles of all these triangles is $2n$ right angles. Let v right angles represent the sum of the angles at the vertex of the polyedral; then, $2n$ right angles being the sum of all the angles of the triangles, $2n - v$ is the sum of the angles at their bases.

Now, at each vertex of the polygon is a triedral having an angle of the polygon for one face, and angles at the bases of the triangles for the other two faces. Then, since two faces of a triedral are greater than the third, the sum of all the angles at the bases of the triangles is greater than the sum of the angles of the polygon. That is,

$$2n - v > 2n - 4.$$

Adding to both members of this inequality, $v + 4$, and subtracting $2n$, we have $4 > v$. That is, the sum of the angles at the vertex is less than four right angles.

This demonstration is a generalization of that of Article 587. The student should make a diagram and special demonstration for a polyedral of five or six faces.

613. Theorem.—*In any convex polyedral, the sum of the diedrals is greater than the sum of the angles of a polygon having the same number of sides that the polyedral has faces.*

Let the given polyedral be divided by diagonal planes into triedrals. Then this theorem may be demonstrated like the analogous proposition on polygons (423). The remark made in Article 346 is also applicable here.

DESCRIPTIVE GEOMETRY.

614. In the former part of this work, we have found problems in drawing to be the best exercises on the principles of Plane Geometry. At first it appears impossible to adapt such problems to the Geometry of Space; for a drawing is made on a plane surface, while the figures here investigated are not plane figures.

This object, however, has been accomplished by the most ingenious methods, invented, in great part, by *Monge*, one of the founders of the Polytechnic School at Paris, the first who reduced to a system the elements of this science, called Descriptive Geometry.

DESCRIPTIVE GEOMETRY is that branch of mathematics which teaches how to represent and determine, by means of drawings on a plane surface, the absolute or relative position of points or magnitudes in space. It is beyond the design of the present work to do more than allude to this interesting and very useful science.

EXERCISES.

615.—1. What is the locus of those points in space, each of which is equally distant from three given points?

2. What is the locus of those points in space, each of which is equally distant from two given planes?

3. What is the locus of those points in space, each of which is equally distant from three given planes?

4. What is the locus of those points in space, each of which is equally distant from two given straight lines which lie in the same plane?

5. What is the locus of those points in space, each of which is equally distant from three given straight lines which lie in the same plane?

6. What is the locus of those points in space, such that the sum of the distances of each from two given planes is equal to a given straight line?

7. If each diedral of a triedral be bisected, the three planes have one common intersection.

8. If a straight line is perpendicular to a plane, every plane parallel to the given line is perpendicular to the given plane.

9. Given any two straight lines in space; either one plane may pass through both, or two parallel planes may pass through them respectively.

10. In the second case of the preceding exercise, a line which is perpendicular to both the given lines is also perpendicular to the two planes.

11. If one face of a triedral is rectangular, then an adjacent diedral angle and its opposite face are either both acute, both right, or both obtuse.

12. Apply to planes, diedrals, and triedrals, respectively, such properties of straight lines, angles, and triangles, as have not already been stated in this chapter, determining, in each case, whether the principle is true when so applied.

CHAPTER X.

POLYEDRONS.

616. A POLYEDRON is a solid, or portion of space, bounded by plane surfaces. Each of these surfaces is a *face*, their several intersections are *edges*, and the points of meeting of the edges are *vertices* of the polyedron.

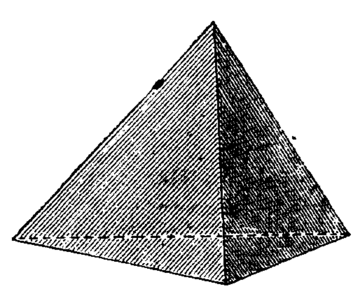
617. Corollary.—The edges being intersections of planes, must be straight lines. It follows that the faces of a polyedron are polygons.

618. A DIAGONAL of a polyedron is a straight line joining two vertices which are not in the same face.

A DIAGONAL PLANE is a plane passing through three vertices which are not in the same face.

TETRAEDRONS.

619. We have seen that three planes can not inclose a space (581). But if any point be taken on each edge of a triedral, a plane passing through these three points would, with the three faces of the triedral, cut off a portion of space, which would be inclosed by four triangular faces.



A TETRAEDRON is a polyedron having four faces.

620. Problem.—*Any four points whatever, which do not all lie in one plane, may be taken as the four vertices of a tetraedron.*

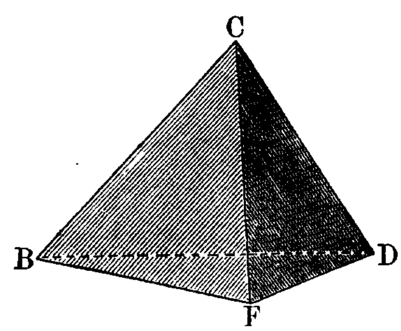
For they may be joined two and two, by straight lines, thus forming the six edges; and these bound the four triangular faces of the figure.

621. Either face of the tetraedron may be taken as the *base.* Then the other faces are called the *sides*, the vertex opposite the base is called the *vertex* of the tetraedron, and the *altitude* is the perpendicular distance from the vertex to the plane of the base. In some cases, the perpendicular falls on the plane of the base produced, as in triangles.

622. Corollary.—If a plane parallel to the base of a tetraedron pass through the vertex, the distance between this plane and the base is the altitude of the tetraedron (574).

623. Theorem.—*There is a point equally distant from the four vertices of any tetraedron.*

In the plane of the face BCF, suppose a circle whose circumference passes through the three points B, C, and F. At the center of this circle, erect a line perpendicular to the plane of BCF.



Every point of this perpendicular is equally distant from the three points B, C, and F. (531).

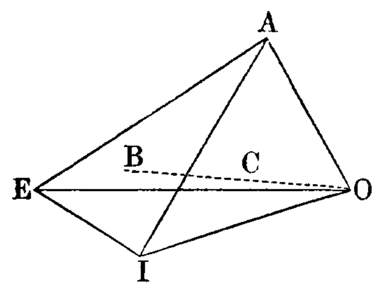
In the same manner, let a line perpendicular to the plane of BDF be erected, so that every point shall be equally distant from the points B, D, and F.

These two perpendiculars both lie in one plane, the plane which bisects the edge BF perpendicularly at its center (520). These two perpendiculars to two oblique planes, being therefore oblique to each other, will meet at some point. This point is equally distant from the four vertices B, C, D, and F.

624. Corollary.—The six planes which bisect perpendicularly the several edges of a tetraedron all meet in one point. But this point is not necessarily within the tetraedron.

625. Theorem.—*There is a point within every tetraedron which is equally distant from the several faces.*

Let AEIO be any tetraedron, and let OB be the straight line formed by the intersection of two planes, one of which bisects the diedral angle whose edge is AO, and the other the diedral whose edge is EO.



Now, every point of the first bisecting plane is equally distant from the faces IAO and EAO (560); and every point of the second bisecting plane is equally distant from the faces EAO and EIO. Therefore, every point of the line BO, which is the intersection of those bisecting planes, is equally distant from those three faces.

Then let a plane bisect the diedral whose edge is EI, and let C be the point where this plane cuts the line BO.

Since every point of this last bisecting plane is equally distant from the faces EAI and EOI, it follows that the point C is equally distant from the four faces of the tetraedron. Since all the bisecting planes are interior, the point found is within the tetraedron.

626. Corollary.—The six planes which bisect the several diedral angles of a tetraedron all meet at one point.

EQUALITY OF TETRAEDRONS.

627. Theorem.—*Two tetraedrons are equal when three faces of the one are respectively equal to three faces of the other, and they are similarly arranged.*

For the three sides of the fourth face, in one, must be equal to the same lines in the other. Hence, the fourth faces are equal. Then each diedral angle in the one is equal to its corresponding diedral angle in the other (599). In a word, every part of the one figure is equal to the corresponding part of the other, and the equal parts are similarly arranged. Therefore, the two tetraedrons are equal.

628. Corollary.—Two tetraedrons are equal when the six edges of the one are respectively equal to those of the other, and they are similarly arranged.

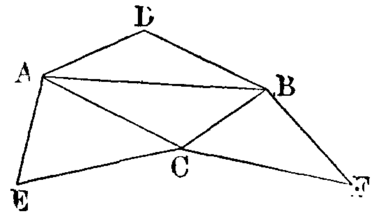
629. Corollary.—Two tetraedrons are equal when two faces and the included diedral of the one are respectively equal to those parts of the other, and they are similarly arranged.

630. Corollary.—Two tetraedrons are equal when one face and the adjacent diedrals of the one are respectively equal to those parts of the other, and they are similarly arranged.

631. When tetraedrons are composed of equal parts in reverse order, they are symmetrical.

MODEL TETRAEDRON.

632. The student may easily construct a model of a tetraedron when the six edges are given. First, with three of the edges which are sides of one face, draw the triangle, as ABC. Then, on each side of this first triangle, as a base, draw a triangle equal to the corresponding face; all of which can be done, for the edges, that is, the sides of these triangles, are given. Then, cut out the whole figure from the paper and carefully fold it at the lines AB, BC, and CA. Since BF is equal to BD, CF to CE, and AD to AE, the points F, D, and E may be united to form a vertex.



In this way models of various forms may be made with more accuracy than in wood, and the student may derive much help from the work.

But he must never forget that the geometrical figure exists only as an intellectual conception. To assist him in this, he should strive to generalize every demonstration, stating the argument without either model or diagram, as in the demonstration last given.

To construct models of symmetrical tetraedrons, the drawings may be equal, but the folding should, in the one case, be up, and in the other, down.

SIMILAR TETRAEDRONS.

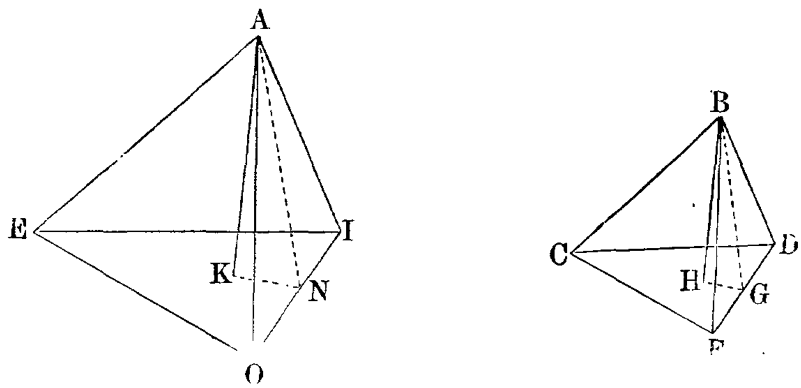
633. Since similarity consists in having the same form, so that every difference of direction in one of two similar figures has its corresponding equal difference of direction in the other, it follows that when two polyedrons are similar, their homologous faces are similar polygons, their homologous edges are of equal diedral angles, and their homologous vertices are of equal polyedrals.

634. Theorem.— *When two tetraedrons are similar, any edge or other line in the one is to the homologous line in the second, as any other line in the first is to its homologous line in the second.*

If the proportion to be proved is between sides of homologous triangles, it follows at once from the similarity of the triangles.

When the edges taken in one of the tetraedrons are not sides of one face; as,

AE : BC : : IO : DF,



then, AE : BC : : IE : CD, as just proved,
and IO : DF : : IE : CD.
Therefore, AE : BC : : IO : DF.

Again, suppose it is to be proved that the altitudes AK and BH have the same ratios as two homologous edges. AK and BH are perpendicular lines let fall from the homologous points A and B on the opposite faces. From K let the perpendicular KN fall upon the edge IO. Join AN, and from H let the perpendicular HG fall upon DF, which is homologous to IO. Join BG.

Now, the planes AKN and EIO are perpendicular to each other (556), and the line IN in one of them is, by construction, perpendicular to their intersection KN. Hence, IN is perpendicular to the plane AKN (557). Therefore, the line AN is perpendicular to IN, and the diedral whose edge is IO is measured by the angle ANK. In the same way, it is proved that the diedral whose edge is DF, is measured by the angle BGH. But these two diedrals, being homologous, are equal, the angles ANK and BGH are equal, and the right angled triangles AKN and BHG are similar. Therefore,

AK : BH : : AN : BG.

Also, the right angled triangles ANI and BGD are similar, since, by hypothesis, the angles AIN and BDG are equal. Hence,

AI : BD :: AN : BG.

Therefore, AK : BH :: AI : BD.

Thus, by the aid of similar triangles, it may be proved that any two homologous lines, in two similar tetraedrons, have the same ratio as two homologous edges.

635. Theorem. — *Two tetraedrons are similar when their faces are respectively similar triangles, and are similarly arranged.*

For we know, from the similarity of the triangles, that every line made on the surface of one may have its homologous line in the second, making angles equal to those made by the first line.

If lines be made through the figure, it may be shown, by the aid of auxiliary lines, as in the corresponding proposition of similar triangles, that every possible angle in the one figure has its homologous equal angle in the other.

The student may draw the diagrams, and go through the details of the demonstration.

636. If the similar faces were not arranged similarly, but in reverse order, the tetraedrons would be *symmetrically similar.*

637. Corollary.—Two tetraedrons are similar when three faces of the one are respectively similar to those of the other, and they are similarly arranged. For the fourth faces, having their sides proportional, are similar also.

638. Corollary.—Two tetraedrons are similar when two triedral vertices of the one are respectively equal to two of the other, and they are similarly arranged.

639. Corollary.—Two tetraedrons are similar when the edges of one are respectively proportional to those of the other, and they are similarly arranged.

640. Theorem.—*The areas of homologous faces of similar tetraedrons are to each other as the squares of their edges.*

This is only a corollary of the theorem that the areas of similar triangles are to each other as the squares of their sides.

641. Corollary.—The areas of homologous faces of similar tetraedrons are to each other as the squares of any homologous lines.

642. Corollary.—The area of any face of one tetraedron is to the area of a homologous face of a similar tetraedron, as the area of any other face of the first is to the area of the homologous face of the second.

643. Corollary.—The area of the entire surface of one tetraedron is to that of a similar tetraedron as the squares of homologous lines.

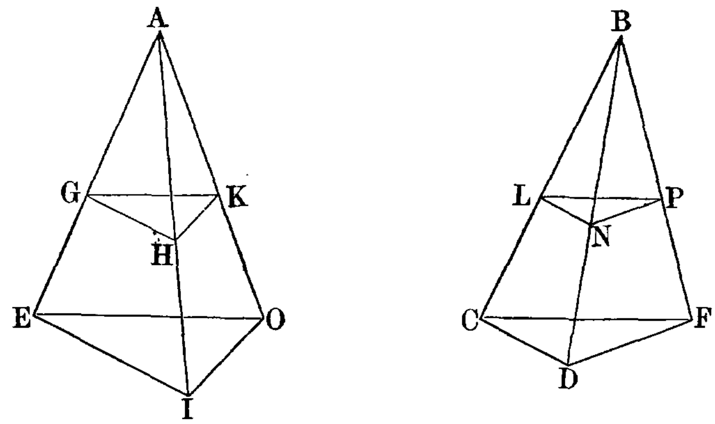
TETRAEDRONS CUT BY A PLANE.

644. Theorem.—*If a plane cut a tetraedron parallel to the base, the tetraedron cut off is similar to the whole.*

For each triangular side is cut by a line parallel to its base (572), thus making all the edges of the two tetraedrons respectively proportional.

645. Theorem.—*If two tetraedrons, having the same altitude and their bases on the same plane, are cut by a plane parallel to their bases, the areas of the sections will have the same ratio as the areas of the bases.*

If a plane parallel to the bases pass through the vertex A, it will also pass through the vertex B (622). But such a plane is parallel to the cutting plane GHP (566).



Therefore, the tetraedrons AGHK and BLNP have equal altitudes.

The tetraedrons AEIO and AGHK are similar (644). Therefore, EIO, the base of the first, is to GHK, the base of the second, as the square of the altitude of the first is to the square of the altitude of the second (641). For a like reason, the base CDF is to the báse LNP as the square of the greater altitude is to the square of the less.

Therefore, EIO : GHK : : CDF : LNP.

By alternation,

EIO : CDF : : GHK : LNP.

646. Corollary.—When the bases are equivalent the sections are equivalent.

647. Corollary.—When the bases are equal the sections are equal. For they are similar and equivalent.

REGULAR TETRAEDRON.

648. There is one form of the tetraedron which deserves particular notice. It has all its faces equilateral. This is called a regular tetraedron.

649. Corollary.—It follows, from the definition, that the faces are equal triangles, the vertices are of equal triedrals, and the edges are of equal diedral angles.

650. The area of the surface of a tetraedron is found by taking the sum of the areas of the four faces. When two or more of them are equal, the process is shortened by multiplication. But the discussion of this matter will be included in the subject of the areas of pyramids.

The investigation of the measures of volumes will be given in another connection.

EXERCISES.

651.—1. State other cases, when two tetraedrons are similar, in addition to those given, Articles 635 to 639.

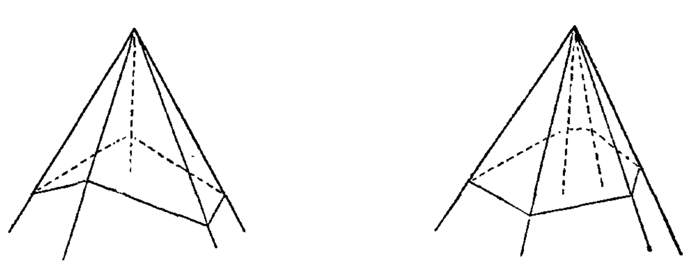
2. In any tetraedron, the lines which join the centers of the opposite edges bisect each other.

3. If one of the vertices of a tetraedron is a trirectangular triedral, the square of the area of the opposite face is equal to the sum of the squares of the areas of the other three faces.

PYRAMIDS.

652. If a polyedral is cut by a plane which cuts its several edges, the section is a polygon, and a portion of space is cut off, which is called a pyramid.

A Pyramid is a polyedron having for one face any



polygon, and for its other faces, triangles whose vertices meet at one point.

The polygon is the *base* of the pyramid, the triangles are its *sides*, and their intersections are the *lateral edges* of the pyramid. The vertex of the polyedral is the *vertex* of the pyramid, and the perpendicular distance from that point to the plane of the base is its *altitude.*

Pyramids are called triangular, quadrangular, pentagonal, etc., according to the polygon which forms the base. The tetraedron is a triangular pyramid.

653. Problem.—*Every pyramid can be divided into the same number of tetraedrons as its base can be into triangles.*

Let a diagonal plane pass through the vertex of the pyramid and each diagonal of the base, and the solution is evident.

EQUAL PYRAMIDS.

654. Theorem.—*Two pyramids are equal when the base and two adjacent sides of the one are respectively equal to the corresponding parts of the other, and they are similarly arranged.*

For the triedrals formed by the given faces in the two must be equal, and may therefore coincide; and the given faces will also coincide, being equal. But now the vertices and bases of the two pyramids coincide. These include the extremities of every edge. Therefore, the edges coincide; also the faces, and the figures throughout.

SIMILAR PYRAMIDS.

655. Theorem.—*Two similar pyramids are composed of tetraedrons respectively similar, and similarly arranged; and, conversely, two pyramids are similar when composed of similar tetraedrons, similarly arranged.*

656. Theorem.—*When a pyramid is cut by a plane parallel to the base, the pyramid cut off is similar to the whole.*

These theorems may be demonstrated by the student. Their demonstration is like that of analogous propositions in triangles and tetraedrons.

REGULAR PYRAMIDS.

657. A REGULAR PYRAMID is one whose base is a regular polygon, and whose vertex is in the line perpendicular to the base at its center.

658. Corollary.—The lateral edges of a regular pyramid are all equal (529), and the sides are equal isosceles triangles.

659. The SLANT HIGHT of a regular pyramid is the perpendicular let fall from the vertex upon one side of the base. It is therefore the altitude of one of the sides of the pyramid.

660. Theorem.—*The area of the lateral surface of a regular pyramid is equal to half the product of the perimeter of the base by the slant hight.*

The area of each side is equal to half the product of its base by its altitude (386). But the altitude of each of the sides is the slant hight of the pyramid, and the sum of all the bases of the sides is the perimeter of the base of the pyramid.

Therefore, the area of the lateral surface of the pyramid, which is the sum of all the sides, is equal to half the product of the perimeter of the base by the slant hight.

661. When a pyramid is cut by a plane parallel to the base, that part of the figure between this plane and the base is called a *frustum* of a pyramid, or a *truncated* pyramid.

662. Corollary.—The sides of a frustum of a pyramid are trapezoids (572); and the sides of the frustum of a regular pyramid are equal trapezoids.

663. The section made by the cutting plane is called the *upper base* of the frustum. The *slant hight* of the frustum of a regular pyramid is that part of the slant hight of the original pyramid which lies between the bases of the frustum. It is therefore the altitude of one of the lateral sides.

664. Theorem.—*The area of the lateral surface of the frustum of a regular pyramid is equal to half the product of the sum of the perimeters of the bases by the slant hight.*

The area of each trapezoidal side is equal to half the product of the sum of its parallel bases by its altitude (392), which is the slant hight of the frustum. Therefore, the area of the lateral surface, which is the sum of all these equal trapezoids, is equal to the product of half the sum of the perimeters of the bases of the frustum, multiplied by the slant hight.

665. Corollary.—The area of the lateral surface of a frustum of a regular pyramid is equal to the product of the perimeter of a section midway between the two bases, multiplied by the slant hight. For the perimeter of a section, midway between the two bases, is equal to half the sum of the perimeters of the bases.

666. Corollary.—The area of the lateral surface of a regular pyramid is equal to the product of the slant hight by the perimeter of a section, midway between the vertex and the base. For the perimeter of the middle section is one-half the perimeter of the base.

MODEL PYRAMIDS.

667. The student may construct a model of a regular pyramid. First, draw a regular polygon of any number of sides. Upon these sides, as bases, draw equal isosceles triangles, taking care that their altitude be greater than the apothem of the base. The figure may then be cut out and folded.

EXERCISES.

668.—1. Find the area of the surface of a regular octagonal pyramid whose slant hight is 5 inches, and a side of whose base is 2 inches.

2. What is the area in square inches of the entire surface of a regular tetraedron, the edge being one inch? *Ans.* $\sqrt{3}$.

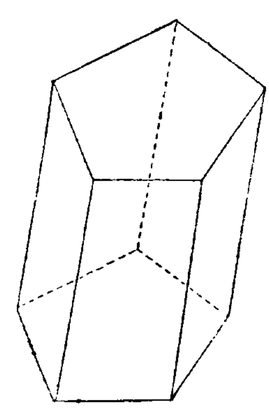
3. A pyramid is regular when its sides are equal isosceles triangles, whose bases form the perimeter of the base of the pyramid.

4. State other cases of equal pyramids, in addition to those given, Article 654.

5. When two pyramids of equal altitude have their bases in the same plane, and are cut by a plane parallel to their bases, the areas of the sections are proportional to the areas of the bases.

PRISMS.

669. A Prism is a polyedron which has two of its faces equal polygons lying in parallel planes, and the other faces parallelograms. Its possibility is shown by supposing two equal and parallel polygons lying in two parallel planes (569). The equal sides being parallel, let planes unite them. The figure thus formed on each plane is a parallelogram, for it has two opposite sides equal and parallel.



The parallel polygons are called the *bases*, the parallelograms the *sides* of the prism, and the intersections of the sides are its *lateral edges*.

The *altitude* of a prism is the perpendicular distance between the planes of its bases.

670. Corollary.—The lateral edges of a prism are all parallel to each other, and therefore equal to each other (573).

671. A Right Prism is one whose lateral edges are perpendicular to the bases.

A Regular Prism is a right prism whose base is a regular polygon.

672. Corollary.—The altitude of a right prism is equal to one of its lateral edges; and the sides of a right prism are rectangles. The sides of a regular prism are equal.

673. Theorem.—*If two parallel planes pass through a prism, so that each plane cuts every lateral edge, the sections made by the two planes are equal polygons.*

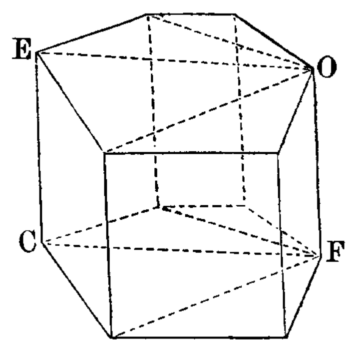
Each side of one of the sections is parallel to the corresponding side of the other section, since they are the intersections of two parallel planes by a third. Hence, that portion of each side of the prism which is between the secant planes, is a parallelogram. Since the sections have their sides respectively equal and parallel, their angles are respectively equal. Therefore, the polygons are equal.

674. Corollary.—The section of a prism made by a plane parallel to the base is equal to the base, and the given prism is divided into two prisms. If two parallel planes cut a prism, as stated in the above theorem, that part of the solid between the two secant planes is also a prism.

HOW DIVISIBLE.

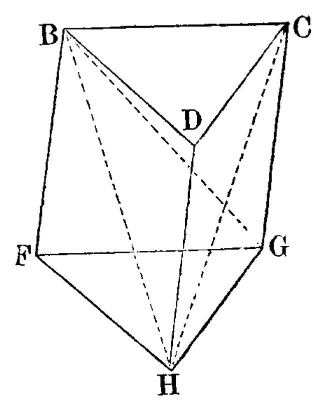
675. Problem.—*Every prism can be divided into the same number of triangular prisms as its base can be into triangles.*

If homologous diagonals be made in the two bases, as EO and CF, they will lie in one plane. For CE and OF being parallel to each other (670), lie in one plane. Therefore, through each pair of these homologous diagonals a plane may pass, and these diagonal planes divide the prisms into triangular prisms.



676. Problem.—*A triangular prism may be divided into three tetraedrons, which, taken two and two, have equal bases and equal altitudes.*

Let a diagonal plane pass through the points B, C, and H, making the intersections BH and CH, in the sides DF and DG. This plane cuts off the tetraedron BCDH, which has for one of its faces the base BCD of the prism; for a second face, the triangle BCH, being the section made by the diagonal plane; and for its other two faces, the triangles BDH and CDH, each being half of one of the sides of the prism.



The remainder of the prism is a quadrangular pyramid, having the parallelogram BCGF for its base, and H for its vertex. Let it be cut by a diagonal plane through the points H, G, and B.

This plane separates two tetraedrons, HBCG and HBFG. The two faces, HBC and HBG, of the tetraedron HBCG, are sections made by the diagonal planes; and the two faces, HCG and BCG, are each half of one side of the prism. The tetraedron HBFG has for one of its faces the base HFG of the prism; for a second face, the triangle HBG, being the section made by the diagonal plane; and, for the other two, the triangles HBF and GBF, each being half of one of the sides of the prism.

Now, consider these two tetraedrons as having their bases BCG and BFG. These are equal triangles lying in one plane. The point H is the common vertex, and therefore they have the same altitude; that is, a perpendicular from H to the plane BCGF.

Next, consider the first and last tetraedrons described, HBCD and BFGH, the former as having BCD for its base, and H for its vertex; the latter as having FGH for its base, and B for its vertex. These bases are equal, being the bases of the given prism. The vertex of each is in the plane of the base of the other. Therefore, the altitudes are equal, being the distance between these two planes.

Lastly, consider the tetraedrons BCDH and BCGH as having their bases CDH and CGH. These are equal triangles lying in one plane. The tetraedrons have the common vertex B, and hence have the same altitude.

677. Corollary.—Any prism may be divided into tetraedrons in several ways; but the methods above explained are the simplest.

678. Remark.—On account of the importance of the above problem in future demonstrations, the student is advised to make a model triangular prism, and divide it into tetraedrons. A potato may be used for this purpose. The student will derive most benefit from those models and diagrams which he makes himself.

EQUAL PRISMS.

679. Theorem.—*Two prisms are equal, when a base and two adjacent sides of the one are respectively equal to the corresponding parts of the other, and they are similarly arranged.*

For the triedrals formed by the given faces in the two prisms must be equal (599), and may therefore be made to coincide. Then the given faces will also coincide, being equal. These coincident points include all of one base, and several points in the second. But the second bases have their sides respectively equal, and parallel to those of the first. Therefore, they also coincide, and the two prisms having both bases coincident, must coincide throughout.

680. Corollary.—Two right prisms are equal when they have equal bases and the same altitude.

681. The theory of similar prisms presents nothing difficult or peculiar. The same is true of symmetrical prisms, and of symmetrically similar prisms.

AREA OF THE SURFACE.

682. Theorem.—*The area of the lateral surface of a prism is equal to the product of one of the lateral edges by the perimeter of a section, made by a plane perpendicular to those edges.*

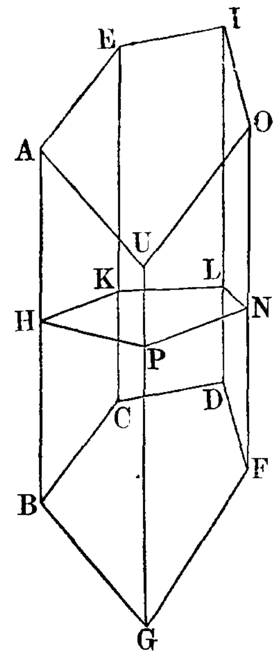
Since the lateral edges are parallel, the plane HN, perpendicular to one, is perpendicular to all of them. Therefore, the sides of the polygon, HK, KL, etc., are severally perpendicular to the edges of the prism which they unite (519).

Then, in order to measure the area of each face of the prism, we take one edge of the prism as the base of the parallelogram, and one side of the polygon HN as its altitude.

Thus,

$$\text{area } AG = AB \times HP,$$

$$\text{area } EB = EC \times HK, \text{ etc.}$$



By addition, the sum of the areas of these parallelograms is the lateral surface of the prism, and the sum of the altitudes of the parallelograms is the perimeter of the polygon HN. Then, since the edges are equal, the area of all the sides is equal to the product of one edge, multiplied by the perimeter of the polygon.

683. Corollary.—The area of the lateral surface of a right prism is equal to the product of the altitude by the perimeter of the base.

684. Corollary.—The area of the entire surface of a regular prism is equal to the product of the perimeter of the base by the sum of the altitude of the prism and the apothem of the base.

EXERCISES.

685.—1. A right prism has less surface than any other prism of equal base and equal altitude; and a regular prism has less surface than any other right prism of equivalent base and equal altitude.

2. A regular pyramid and a regular prism have equal hexagonal bases, and altitudes equal to three times the radius of the base; required the ratio of the areas of their lateral surfaces.

3. Demonstrate the principle stated in Article 683, without the aid of Article 682.

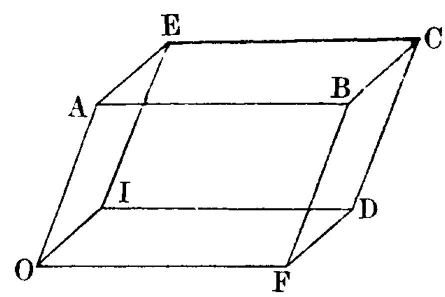
MEASURE OF VOLUME.

686. A PARALLELOPIPED is a prism whose bases are parallelograms. Hence, a parallelopiped is a solid inclosed by six parallelograms.

687. Theorem.—*The opposite sides of a parallelopiped are equal.*

For example, the faces AI and BD are equal.

For IO and DF are equal, being opposite sides of the parallelogram IF. For a like reason, EI is equal to CD. But, since these equal sides are also parallel, the included angles EIO and CDF are equal. Hence, the parallelograms are equal.



688. Corollary.—Any two opposite faces of a parallelopiped may be assumed as the bases of the figure.

689. A parallelopiped is called *right* in the same case as any other prism. When the bases also are rectangles, it is called *rectangular*. Then all the faces are rectangles.

690. A CUBE is a rectangular parallelopiped whose length, breadth, and altitude are equal. Then a cube is a solid, bounded by six equal squares. All its vertices, being trirectangular triedrals, are equal (602). All its edges are of right diedral angles, and therefore equal (555).

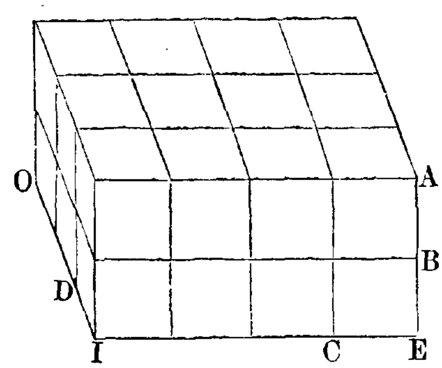
The cube has the simplest form of all geometrical solids. It holds the same rank among them that the square does among plane figures, and the straight line among lines.

The cube is taken, therefore, as the unit of measure of volume. That is, whatever straight line is taken as the unit of length, the cube whose edge is of that length is the unit of volume, as the square whose side is of that length is the measure of area.

VOLUME OF PARALLELOPIPEDS.

691. Theorem.—*The volume of a rectangular parallelopiped is equal to the product of its length, breadth, and altitude.*

In the measure of the rectangle, the product of one line by another was explained. Here we have three lines used with a similar meaning. That is, the number of cubical units contained in a rectangular parallelopiped is equal to the product of the numbers of linear units in the length, the breadth, and the altitude.



If the altitude AE, the length EI, and the breadth IO, have a common measure, let each be divided by it; and let planes, parallel to the faces of the prism, pass through all the points of division, B, C, D, etc.

By this construction, all the angles formed by these planes and their intersections are right angles, and each of the intercepted lines is equal to the linear unit used in dividing the edges of the prism. Therefore, the prism is divided into equal cubes. The number of these at the base is equal to the number of rows, multiplied by the number in each row; that is, the product of the length by the breadth. There are as many layers of cubes as there are linear units of altitude. Therefore, the whole number is equal to the product of the length, breadth, and altitude. In the diagram, the dimensions being four, three, and two, the volume is twenty-four.

But if the length, breadth, and altitude have no common measure, a linear unit may be taken, successively smaller and smaller. In this, we would not take the whole of the linear dimensions, nor would we measure the whole of the prism. But the remainder of both would grow less and less. The part of the prism measured at each step, would be measured exactly by the principle just demonstrated.

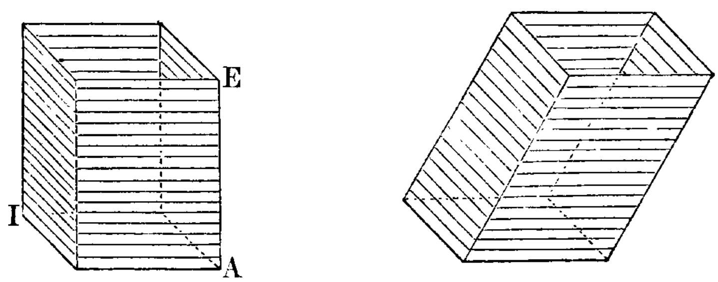
By these successive diminutions of the unit, we can make the part measured approach to the whole prism as nearly as we please. In a word, the whole is the limit of the parts measured; and since the principle demonstrated is true up to the limit, it must be true at the limit. Therefore, the rectangular parallelopiped is measured by the product of its length, breadth, and altitude.

692. Theorem.—*The volume of any parallelopiped is equal to the product of its length, breadth, and altitude.*

Inasmuch as this has just been demonstrated for the rectangular parallelopiped, it will be sufficient to show that any parallelopiped is equivalent to a rectangular one having the same linear dimensions.

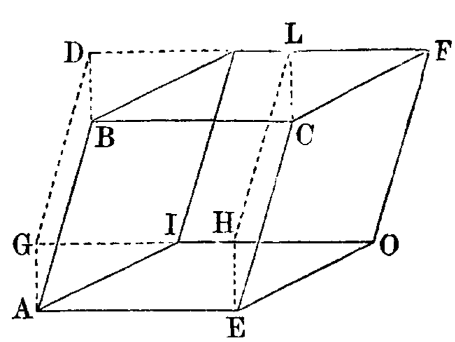
Suppose the lower bases of the two prisms to be placed on the same plane. Then their upper bases must also be in one plane, since they have the same altitude. Let the altitude AE be divided into an infinite number of equal parts, and through each point of division pass a plane parallel to the base AI.

Now, every section in either prism is equal to the base; but the bases of the two prisms, having the same length and breadth, are equivalent. The several partial infinitesimal prisms are reduced to equivalent fig-



ures. Although they are not, strictly speaking, parallelograms, yet their altitudes being infinitesimal, there can be no error in considering them as plane figures; which, being equal to their respective bases, are equivalent. Then, the number of these is the same in each prism. Therefore, the sum of the whole, in one, is equivalent to the sum of the whole, in the other; that is, the two parallelopipeds are equivalent.

Besides the above demonstration by the method of infinites, the theorem may be demonstrated by the ordinary method of reasoning, which is deduced from principles that depend upon the superposition and coincidence of equal figures, as follows.



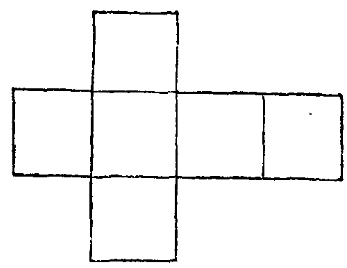
Let AF be any oblique parallelopiped. It may be shown to be equivalent to the parallelopiped AL, which has a rectangular base, AH, since the prism LHEO is equal to the prism DGAI. But the parallelopipeds AF and AL have the same length, breadth, and altitude.

By similar reasoning, the prism AL may be shown to be equivalent to a prism of the same base and altitude, but with two of its opposite sides rectangular. This third prism may then be shown to be equivalent to a fourth, which is rectangular, and has the same dimensions as the others.

693. Corollary.—The volume of a cube is equal to the third power of its edge. Thence comes the name of cube, to designate the third power of a number.

MODEL CUBES.

694. Draw six equal squares, as in the diagram. Cut out the figure, fold at the dividing lines, and glue the edges. It is well to have at least eight of one size.



695. Corollary.—The volume of any parallelopiped is equal to the product of its base by its altitude.

696. Corollary.—The volumes of any two parallelopipeds are to each other as the products of their three dimensions.

VOLUME OF PRISMS.

697. Theorem.—*The volume of any triangular prism is equal to the product of its base by its altitude.*

The base of any right triangular prism may be considered as one-half of the base of a right parallelopiped. Then the whole parallelopiped is double the given prism, for it is composed of two right prisms having equal bases and the same altitude, of which the given prism is one. Therefore, the given prism is measured by half the product of its altitude by the base of the parallelopiped; that is, by the product of its own base and altitude.

If the given prism be oblique, it may be shown, by demonstrations similar to the first of those in Article 692, to be equivalent to a right prism having the same base and altitude.

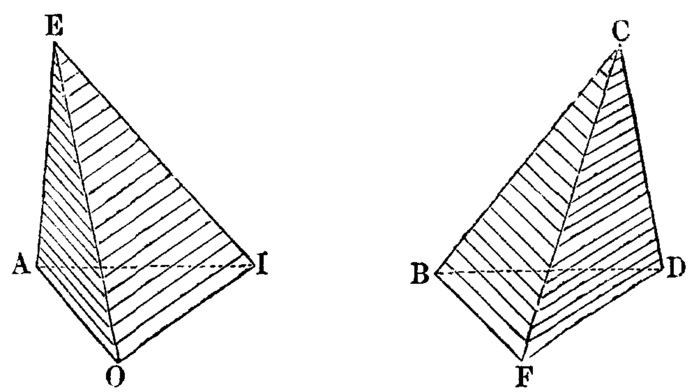
698. Corollary.—The volume of any prism is equal to the product of its base by its altitude. For any prism is composed of triangular prisms, having the common altitude of the given prism, and the sum of their bases forming the given base.

699. Corollary.—The volume of a triangular prism is equal to the product of one of its lateral edges multiplied by the area of a section perpendicular to that edge.

VOLUME OF TETRAEDRONS.

700. Theorem.—*Two tetraedrons of equivalent bases and of the same altitude are equivalent.*

Suppose the bases of the two tetraedrons to be in the



same plane. Then their vertices lie in a plane parallel to the bases, since the altitudes are equal. Let the edge AE be divided into an infinite number of parts, and through each point of division pass a plane parallel to the base AIO.

Now, the several infinitesimal frustums into which the two figures are divided may, without error, be considered as plane figures, since their altitudes are infinitesimal. But each section of one tetraedron is equivalent to the section made by the same plane in the other tetraedron. Therefore, the sum of all the infinitesimal frustums in the one figure is equivalent to the sum of all in the other; that is, the two tetraedrons are equivalent.

701. Theorem.—*The volume of a tetraedron is equal to one-third of the product of the base by the altitude.*

Upon the base of any given tetraedron, a triangular prism may be erected, which shall have the same altitude, and one edge coincident with an edge of the tetraedron. This prism may be divided into three tetraedrons, the given one and two others, which, taken two and two, have equal bases and altitudes (676).

Then, these three tetraedrons are equivalent (700); and the volume of the given tetraedron is one-third of the volume of the prism; that is, one-third of the product of its base by its altitude.

VOLUME OF PYRAMIDS.

702. Corollary.—The volume of any pyramid is equal to one-third of the product of its base by its altitude. For any pyramid is composed of triangular pyramids; that is, of tetraedrons having the common altitude of the given pyramid, and the sum of their bases forming the given base (653).

703. Corollary.—The volumes of two prisms of equivalent bases are to each other as their altitudes, and the volumes of two prisms of equal altitudes are to each other as their bases. The same is true of pyramids.

704. Corollary.—Symmetrical prisms are equivalent. The same is true of symmetrical pyramids.

705. The volume of a frustum of a pyramid is found by subtracting the volume of the pyramid cut off from the volume of the whole. When the altitude of the whole is not given, it may be found by this proportion: the area of the lower base of the frustum is to the area of its upper base, which is the base of the part cut off, as the square of the whole altitude is to the square of the altitude of the part cut off.

EXERCISES.

706.—1. What is the ratio of the volumes of a pyramid and prism having the same base and altitude?

2. If two tetraedrons have a triedral vertex in each equal, their volumes are in the ratio of the products of the edges which contain the equal vertices.

3. The plane which bisects a diedral angle of a tetraedron, divides the opposite edge in the ratio of the areas of the adjacent faces.

SIMILAR POLYEDRONS.

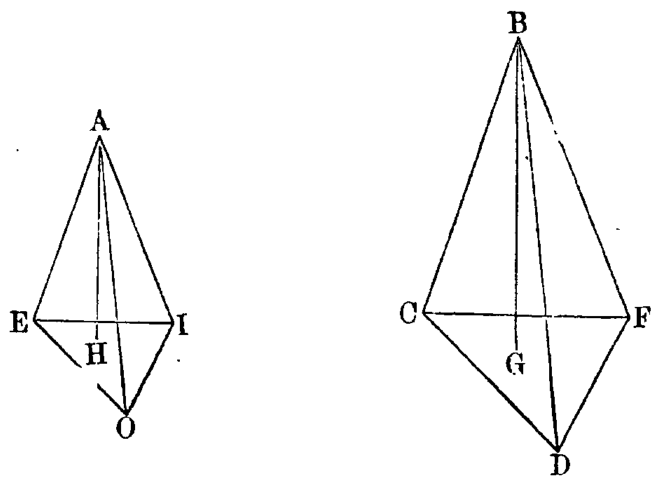
707. The propositions (640 to 643) upon the ratios of the areas of the surfaces of similar tetraedrons, may be applied by the student to any similar polyedrons. These propositions and the following are equally applicable to polyedrons that are symmetrically similar.

708. Problem.—*Any two similar polyedrons may be divided into the same number of similar tetraedrons, which shall be respectively similar, and similarly arranged.*

For, after dividing one into tetraedrons, the construction of the homologous lines in the other will divide it in the same manner. Then the similarity of the respective tetraedrons follows from the proportionality of the lines.

709. Theorem.—*The volumes of similar polyedrons are proportional to the cubes of homologous lines.*

First, suppose the figures to be tetraedrons. Let AH and BG be the altitudes.



Then (641), EIO : CDF : : EI^2 : CF^2 . : AH^2 : BG^2.

By the proportionality of homologous lines, (634),

$\frac{1}{3}$ AH : $\frac{1}{3}$ BG : : EI : CF : : AH : BG.

Multiplying these proportions (701), we have

AEIO : BCFD : : EI^3 : CF^3 : : AH^3 : BG^3,

or, as the cubes of any other homologous lines.

Next, let any two similar polyedrons be divided into the same number of tetraedrons. Then, as just proved, the volumes of the homologous parts are proportional to the cubes of the homologous lines. By arranging these in a continued proportion, as in Article 436, we may show that the volume of either polyedron is to the volume of the other as the cube of any line of the first is to the cube of the homologous line of the second.

710. Notice that in the measure of every area there are two linear dimensions; and in the measure of every volume, three linear, or one linear and one superficial.

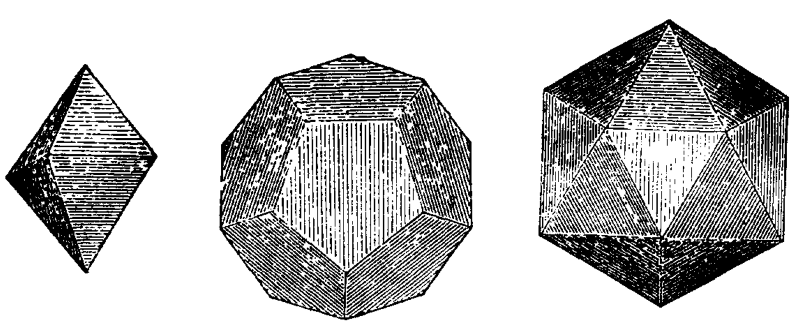
EXERCISE.

711. What is the ratio between the edges of two cubes, one of which has twice the volume of the other?

This problem of the duplication of the cube was one of the celebrated problems of ancient times. It is said that the oracle of Apollo at Delphos, demanded of the Athenians a new altar, of the same shape, but of twice the volume of the old one. The efforts of the Greek geometers were chiefly aimed at a graphic solution; that is, the edge of one cube being given, to draw a line equal to the edge of the other, using no instruments but the rule and compasses. In this they failed. The student will find no difficulty in making an arithmetical solution, within any desired degree of approximation.

REGULAR POLYEDRONS.

712. A REGULAR POLYEDRON is one whose faces are equal and regular polygons, and whose vertices are equal polyedrals.



The regular tetraedron and the cube, or regular hexaedron, have been described.

The regular *octaedron* has eight, the *dodecaedron* twelve, and the *icosaedron* twenty faces.

The class of figures here defined must not be confounded with regular pyramids or prisms.

713. Problem.—*It is not possible to make more than five regular polyedrons.*

First, consider those whose faces are triangles. Each angle of a regular triangle is one-third of two right angles. Either three, four, or five of these may be joined to form one polyedral vertex, the sum being, in each case, less than four right angles (612). But the sum of six such angles is not less than four right angles. Therefore, there can not be more than three kinds of regular polyedrons whose faces are triangles, viz.: the tetraedron, where three plane angles form a vertex; the octaedron, where four, and the icosaedron, where five angles form a vertex.

The same kind of reasoning shows that only one regular polyedron is possible with square faces, the cube; and only one with pentagonal faces, the dodecaedron.

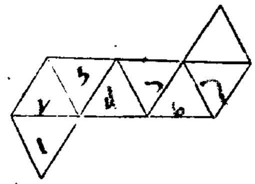
Regular hexagons can not form the faces of a regular polyedron, for three of the angles of a regular hexagon are together not less than four right angles; and therefore they can not form a vertex.

So much the more, if the polygon has a greater number of sides, it will be impossible for its angles to be the faces of a polyedral. Therefore, no polyedron is possible, except the five that have been described.

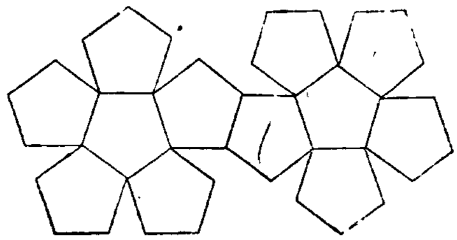
MODEL REGULAR POLYEDRONS.

714. The possibility of regular polyedrons of eight, of twelve, and of twenty sides is here assumed, as the demonstration would occupy more space than the principle is worth. However, the student may construct models of these as follows. Plans for the regular tetraedron and the cube have already been given.

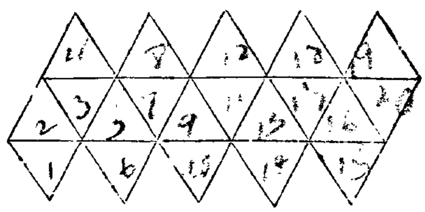
For the octaedron, draw eight equal regular triangles, as in the diagram.



For the dodecaedron, draw twelve equal regular pentagons, as in the diagram.



For the icosaedron, draw twenty equal regular triangles, as in the diagram.



There are many crystals, which, though not regular, in the geometrical rigor of the word, yet present a certain regularity of shape.

EXERCISES.

715.—1. How many edges and how many vertices has each of the regular polyedrons?

2. Calling that point the *center of a triangle* which is the intersection of straight lines from each vertex to the center of the opposite side; then, demonstrate that the four lines which join the vertices of a tetraedron to the centers of the opposite faces, intersect each other in one point.

3. In what ratio do the lines just described in the tetraedron divide each other?

4. The opposite vertices of a parallelopiped are symmetrical triedrals.

5 The diagonals of a parallelopiped bisect each other; the lines which join the centers of the opposite edges bisect each other; the lines which join the centers of the opposite faces bi-

sect each other; and the point of intersection is the same for all these lines.

6. The diagonals of a rectangular parallelopiped are equal.

7. The square of the diagonal of a rectangular parallelopiped is equivalent to the sum of the squares of its length, breadth, and altitude.

8. A cube is the largest parallelopiped of the same extent of surface.

9. If a right prism is symmetrical to another, they are equal.

10. Within any regular polyedron there is a point equally distant from all the faces, and also from all the vertices.

11. Two regular polyedrons of the same number of faces are similar.

12. Any regular polyedron may be divided into as many regular and equal pyramids as it has faces.

13. Two different tetraedrons, and only two, may be formed with the same four triangular faces; and these two tetraedrons are symmetrical.

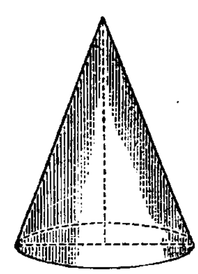
14. The area of the lower base of a frustum of a pyramid is five square feet, of the upper base one and four-fifths square feet, and the altitude is two feet; required the volume.

CHAPTER XI.

SOLIDS OF REVOLUTION.

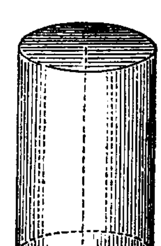
716. Of the infinite variety of forms there remain but three to be considered in this elementary work. These are formed or generated by the revolution of a plane figure about one of its lines as an axis. Figures formed in this way are called *solids of revolution.*

717. A Cone is a solid formed by the revolution of a right angled triangle about one of its legs as an axis. The other leg revolving describes a plane surface (521). This surface is also a circle, having for its radius the leg by which it is described. The hypotenuse describes a curved surface.



The plane surface of a cone is called its *base.* The opposite extremity of the axis is the *vertex.* The *altitude* is the distance from the vertex to the base, and the *slant hight* is the distance from the vertex to the circumference of the base.

718. A Cylinder is a solid described by the revolution of a rectangle about one of its sides as an axis. As in the cone, the sides adjacent to the axis describe circles, while the opposite side describes a curved surface.



The plane surfaces of a cylinder are called its *bases,* and the perpendicular distance between them is its *altitude*.

These figures are strictly a regular cone and a regular cylinder, yet but one word is used to denote the figures defined, since other cones and cylinders are not usually discussed in Elementary Geometry. The sphere, which is described by the revolution of a semicircle about the diameter, will be considered separately.

719. As the curved surfaces of the cone and of the cylinder are generated by the motion of a straight line, it follows that each of these surfaces is straight in one direction.

A straight line from the vertex of the cone to the circumference of the base, must lie wholly in the surface. So a straight line, perpendicular to the base of a cylinder at its circumference, must lie wholly in the surface. For, in each case, these positions had been occupied by the generating lines.

One surface is *tangent* to another when it meets, but being produced does not cut it. The place of contact of a plane with a conical or cylindrical surface, must be a straight line; since, from any point of one of those surfaces, it is straight in one direction.

CONIC SECTIONS.

720. Every point of the line which describes the curved surface of a cone, or of a cylinder, moves in a plane parallel to the base (565). Therefore, if a cone or a cylinder be cut by a plane parallel to the base, the section is a circle.

If we conceive a cone to be cut by a plane, the curve formed by the intersection will be different according to the position of the cutting plane. There are three different modes in which it is possible for the intersection to take place. The curves thus formed are the ellipse, parabola, and hyperbola.

These *Conic Sections* are not usually considered in Elementary Geometry, as their properties can be better investigated by the application of algebra.

CONES.

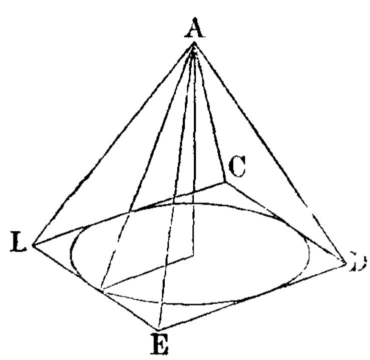
721. A cone is said to be *inscribed* in a pyramid, when their bases lie in one plane, and the sides of the pyramid are tangent to the curved surface of the cone. The pyramid is said to be *circumscribed* about the cone.

A cone is said to be *circumscribed* about a pyramid, when their bases lie in one plane, and the lateral edges of the pyramid lie in the curved surface of the cone. Then the pyramid is *inscribed* in the cone.

722. Theorem.—*A cone is the limit of the pyramids which can be circumscribed about it; also of the pyramids which can be inscribed in it.*

Let ABCDE be any pyramid circumscribed about a cone.

The base of the cone is a circle inscribed in the base of the pyramid. The sides of the pyramid are tangent to the surface of the cone.



Now, about the base of the cone there may be described a polygon of double the number of sides of the first, each alternate side of the second polygon coinciding with a side of the first. This second polygon may be the base of a pyramid, having its vertex at A. Since the sides of its bases are tangent to the base of the cone, every side of the pyramid is tangent to the curved surface of the cone. Thus the second pyramid is circumscribed about the cone, but is itself within the first pyramid.

By increasing the number of sides of the pyramid, it can be made to approximate to the cone within less than any appreciable difference. Then, as the base of the cone is the limit of the bases of the pyramids, the cone itself is also the limit of the pyramids.

Again, let a polygon be inscribed in the base of the cone. Then, straight lines joining its vertices with the vertex of the cone form the lateral edges of an inscribed pyramid. The number of sides of the base of the pyramid, and of the pyramid also, may be increased at will. It is evident, therefore, that the cone is the limit of pyramids, either circumscribed or inscribed.

723. Corollary.—The area of the curved surface of a cone is equal to one-half the product of the slant hight by the circumference of the base (660). Also, it is equal to the product of the slant hight by the circumference of a section midway between the vertex and the base (666).

724. Corollary.—The area of the entire surface of a cone is equal to half of the product of the circumference of the base by the sum of the slant hight and the radius of the base (499).

725. Corollary.—The volume of a cone is equal to one-third of the product of the base by the altitude.

726. The frustum of a cone is defined in the same way as the frustum of a pyramid.

727. Corollary.—The area of the curved surface of the frustum of a cone is equal to half the product of its slant hight by the sum of the circumferences of its bases (664). Also, it is equal to the product of its slant hight by the circumference of a section midway between the two bases (665).

728. Corollary.—If a cone be cut by a plane parallel to the base, the cone cut off is similar to the whole (656).

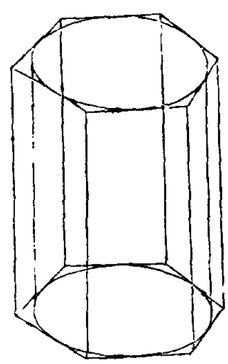
EXERCISES.

729.—1. Two cones are similar when they are generated by similar triangles, homologous sides being used for the axes.

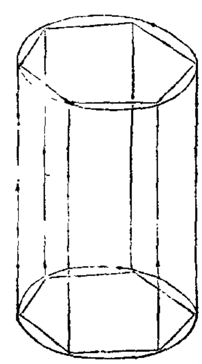
2. A section of a cone by a plane passing through the vertex, is an isosceles triangle.

CYLINDERS.

730. A cylinder is said to be *inscribed* in a prism, when their bases lie in the same planes, and the sides of the prism are tangent to the curved surface of the cylinder. The prism is then said to be *circumscribed* about the cylinder.



A cylinder is said to be *circumscribed* about a prism, when their bases lie in the same planes, and the lateral edges of the prism lie in the curved surface of the cylinder; and the prism is then said to be *inscribed* in the cylinder.



731. Theorem.—*A cylinder is the limit of the prisms which can be circumscribed about it; also of those which can be inscribed in it.*

The demonstration of this theorem is so similar to that of the last, that it need not be repeated.

732. Corollary.—The area of the curved surface of a cylinder is equal to the product of the altitude by the circumference of the base (683).

733. Corollary.—The area of the entire surface of a cylinder is equal to the product of the circumference of the base by the sum of the altitude and the radius of the base (684).

734. Corollary.—The volume of a cylinder is equal to the product of the base by the altitude (698).

MODEL CONES AND CYLINDERS.

735. Models of cones and cylinders may be made from paper, taking a sector of a circle for the curved surface of a cone, and a rectangle for the curved surface of a cylinder. Make the bases separately.

EXERCISES.

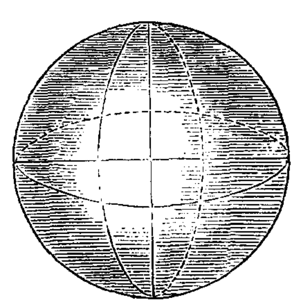
736.—1. Apply to cones and cylinders the principles demonstrated of similar polyedrons.

2. A section of a cylinder made by a plane perpendicular to the base is a rectangle.

3. The axis of a cone or of a cylinder is equal to its altitude.

SPHERES.

737. A SPHERE is a solid described by the revolution of a semicircle about its diameter as an axis.



The *center*, *radius*, and *diameter* of the sphere are the same as those of the generating circle. The spherical surface is described by the circumference.

738. Corollary.—Every point on the surface of the sphere is equally distant from the center.

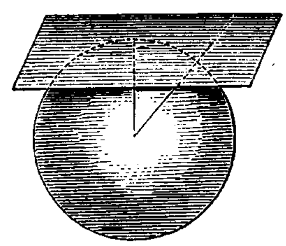
This property of the sphere is frequently given as its definition.

739. Corollary.—All radii of the same sphere are equal. The same is true of the diameters.

740. Corollary.—Spheres having equal radii are equal.

741. Corollary.—A plane passing through the center of a sphere divides it into equal parts. The halves of a sphere are called *hemispheres.*

742. Theorem.—*A plane which is perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*



For if straight lines extend from the center of the sphere to any other point of the plane, they are oblique and longer than the radius, which is perpendicular (530). Therefore, every point of the plane except one is beyond the surface of the sphere, and the plane is tangent.

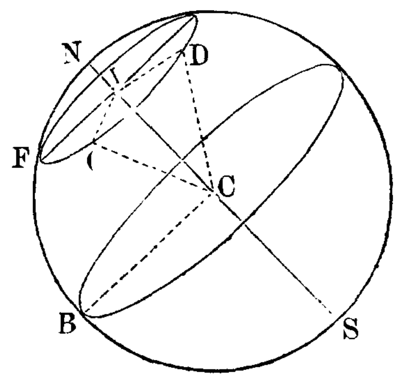
743. Corollary.—The spherical surface is curved in every direction. Unlike those surfaces which are generated by the motion of a straight line, every possible section of it is a curve.

SECANT PLANES.

744. Theorem.– *Every section of a sphere made by a plane is a circle.*

If the plane pass through the center of the sphere, every point in the perimeter of the section is equally distant from the center, and therefore the section is a circle.

But if the section do not pass through the center, as DGF, then from the center C let CI fall perpendicularly on the cutting plane. Let radii of the sphere, as CD and CG, extend to different points of the boundary of the section, and join ID and IG.



Now the oblique lines CD and CG being equal, the points D and G must be equally distant from I, the foot of the perpendicular (529). The same is true of all the points of the perimeter DGF. Therefore, DGF is the circumference of a circle of which I is the center.

745. Corollary.—The circle formed by the section through the center is larger than one formed by any plane not through the center. For the radius BC is equal to GC, and longer than GI (104).

746. When the plane passes through the center of a sphere, the section is called a *great circle;* otherwise it is called a *small circle.*

747. Corollary.—All great circles of the same sphere are equal.

748. Corollary.—Two great circles bisect each other, and their intersection is a diameter of the sphere.

749. Corollary.—If a perpendicular be let fall from the center of a sphere on the plane of a small circle, the foot of the perpendicular is the center of the circle; and conversely, the axis of any circle is a diameter of the sphere.

The two points where the axis of a circle pierces the spherical surface, are the *poles* of the circle. Thus, N and S are the poles of both the sections in the last diagram.

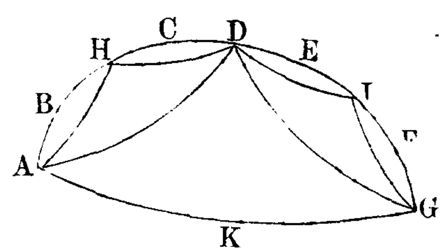
750. Corollary.—Circles whose planes are parallel to each other have the same axis and the same poles.

ARC OF A GREAT CIRCLE.

751. Theorem.—*The shortest line which can extend from one point to another along the surface of a sphere, is the arc of a great circle, passing through the two points.*

Only one great circle can pass through two given points on the surface of a sphere; for these two points and the center determine the position of the plane of the circle.

Let ABCDEFG be any curve whatever on the surface of a sphere from G to A. Let AKG be the arc of a great circle joining these points, and also AD and DG arcs of great circles joining those points with the point D of the given curve.



Then the sum of AD and DG is greater than AKG.

For the planes of these arcs form a triedral whose vertex is at the center of the sphere. These arcs have the same ratios to each other as the plane angles which compose this triedral, for the arcs are intercepted by the sides of the angles, and they have the same radius. But any one of these angles is less than the sum of the other two (586). Therefore, any one of the arcs is less than the sum of the other two.

Again, let AH and HD be arcs of great circles joining A and D with some point H of the given curve; also let DI and IG be arcs of great circles. In the same manner as above, it may be shown that AH and HD are greater than AD, and that the sum of DI and IG is greater than DG. Therefore, the sum of AH, HD, DI, and IG is still greater than AKG.

By continuing to take intermediate points and joining them to the preceding, a series of lines is formed, each greater than the preceding, and each approaching nearer to the given curve. Evidently, this approach can be made as nearly as we choose. Therefore, the curve is the limit of these lines, and partakes of their common character, in being greater than the arc of a great circle which joins its extremities.

752. Theorem.—*Every plane passing through the axis of a circle is perpendicular to the plane of that circle, and its section is a great circle.*

The first part of this theorem is a corollary of Article 556. The second part is proved by the fact that every axis passes through the center of a sphere (749).

753. Corollary.—The distances on the spherical surface from any points of a circumference to its pole, are the same. For the arcs of great circles which mark these distances are equal, since all their chords are equal oblique lines (529).

754. Corollary.—The distance of the pole of a great circle from any point of the circumference is a quadrant.

APPLICATIONS.

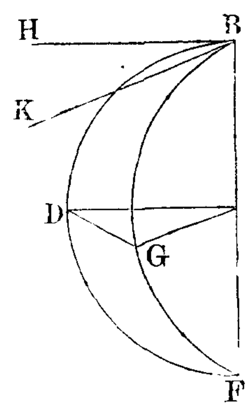
755. The student of geography will recognize the equator as a great circle of the earth, which is nearly a sphere. The parallels of latitude are small circles, all having the same poles as the equator. The meridians are great circles perpendicular to the equator.

The application of the principle of Article 751 to navigation has been one of the greatest reforms in that art. A vessel crossing the ocean from a port in a certain latitude to a port in the same latitude, should not sail along a parallel of latitude, for that is the arc of a small circle.

756. The curvature of the sphere in every direction, renders it impossible to construct an exact model with plane paper. But the student is advised to procure or make a globe, upon which he can draw the diagrams of all the figures. This is the more important on account of the difficulty of clearly representing these figures by diagrams on a plane surface.

SPHERICAL ANGLES.

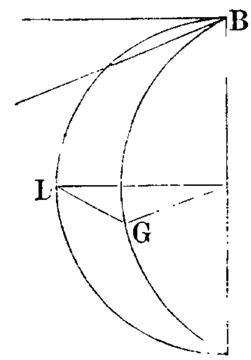
757. A SPHERICAL ANGLE is the difference in the directions of two arcs of great circles at their point of meeting. To obtain a more exact idea of this angle, notice that the direction of an arc at a given point is the same as the direction of a straight line tangent to the arc at that point. Thus, the direction of the arc BDF at the point B, is the same as the direction of the tangent BH.



758. Corollary.—A spherical angle is the same as the plane angle formed by lines tangent to the given arcs at their point of meeting. Thus, the spherical angle DBG is the same as the plane angle HBK, the lines HB and BK being severally tangent to the arcs BD and BG.

759. Corollary.—A spherical angle is the same as the diedral angle formed by the planes of the two arcs. For, since the intersection BF of the planes of the arcs is a diameter (748), the tangents HB and KB are both perpendicular to it, and their angle measures the diedral.

760. Corollary.—A spherical angle is measured by the arc of a circle included between the sides of the angle, the pole of the arc being at the vertex.



Thus, if DG is an arc of a great circle whose pole is at B, then the spherical angle DBG is measured by the arc DG.

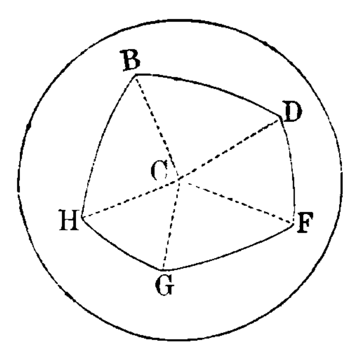
761. A LUNE is that portion of the surface of a sphere included between two halves of great circles.

That portion of the sphere included between the two planes is called a *spherical wedge.* Hence, two great circles divide the surface into four lunes, and the sphere into four wedges.

SPHERICAL POLYGONS.

762. A SPHERICAL POLYGON is that portion of the surface of a sphere included between three or more arcs of great circles.

Let C be the center of a sphere, and also the vertex of a convex polyedral. Then, the planes of the faces of this polyedral will cut the surface of the sphere in arcs of great circles, which form the polygon BDFGH. We say *convex*, for only those polygons which have all the angles convex are considered among spherical polygons. Conversely, if a spherical polygon have the planes of its several sides produced, they form a polyedral whose vertex is at the center of the sphere.



The angles of the polygon are the same as the diedral angles of the polyedral (759).

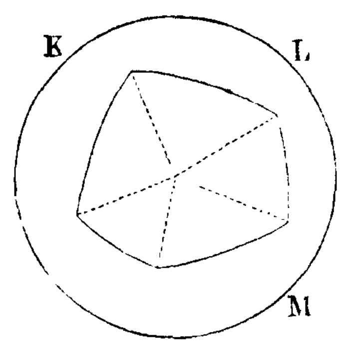
763. Theorem.—*The sum of all the sides of a spherical polygon is less than a circumference of a great circle.*

The arcs which form the sides of the polygon measure the angles which form the faces of the corresponding polyedral, for all the arcs have the same radius.

But the sum of all the faces of the polyedral being less than four right angles, the sum of the sides must be less than a circumference.

764. Theorem.—*A spherical polygon is always within the surface of a hemisphere.*

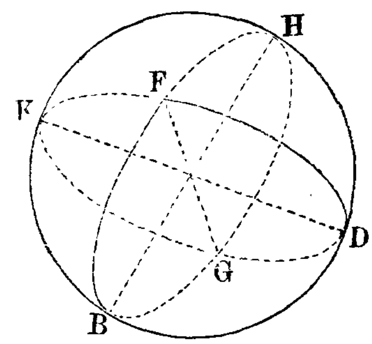
For a plane may pass through the vertex of the corresponding polyedral, having all of the polyedral on one side of it (609). The section formed by this plane produced is a great circle, as KLM. But since the polyedral is on one side of this plane, the corresponding polygon must be contained within the surface on one side of it.



765. That portion of a sphere which is included between a spherical polygon and its corresponding polyedral is called a *spherical pyramid*, the polygon being its base.

SPHERICAL TRIANGLES.

766. If the three planes which form a triedral at the center of a sphere be produced, they divide the sphere into eight parts or spherical pyramids, each having its triedral at the center, and its spherical triangle at the surface. Thus, for every spherical triangle, there are seven others whose sides are respectively either equal or supplementary to those of the given triangle.



Of these seven spherical triangles, that which lies vertically opposite the given triangle, as GKH to FDB, has its sides respectively equal to the sides of the given triangle, but they are arranged in reverse order; for the corresponding triedrals are symmetrical. Such spherical triangles are called *symmetrical.*

767. Corollary.—If two spherical triangles are equal, their corresponding triedrals are also equal; and if two spherical triangles are symmetrical, their corresponding triedrals are symmetrical.

768. Corollary.—On the same sphere, or on equal spheres, equal triedrals at the center have equal corresponding spherical triangles; and symmetrical triedrals at the center have symmetrical corresponding spherical triangles.

769. Corollary.—The three sides and the three angles of a spherical triangle are respectively the measures of the three faces and the three diedrals of the triedral at the center.

770. Corollary.—Spherical triangles are isosceles, equilateral, rectangular, birectangular, and trirectangular, according to their triedrals.

771. Corollary.—The sum of the angles of a spherical triangle is greater than two, and less than six right angles (591).

772. Corollary.—An isosceles spherical triangle is equal to its symmetrical, and has equal angles opposite the equal sides (594).

773. Corollary.—The radius being the same, two spherical triangles are equal,

1st. When they have two sides and the included angle of the one respectively equal to those parts of the other, and similarly arranged;

2d. When they have one side and the adjacent angles of the one respectively equal to those parts of the other, and similarly arranged;

3d. When the three sides are respectively equal, and similarly arranged;

4th. When the three angles are respectively equal, and similarly arranged.

774. Corollary.—In each of the four cases just given, when the arrangement of the parts is reversed, the triangles are symmetrical.

POLAR TRIANGLES.

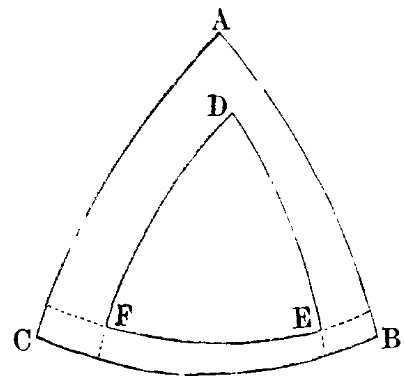
775. If at the vertex of a triedral, a perpendicular be erected to each face, these lines form the edges of a supplementary triedral (590). If the given vertex is at the center of a sphere, then there are two spherical triangles corresponding to these two triedrals, and they have all those relations which have been demonstrated concerning supplementary triedrals.

Since each edge of one triedral is perpendicular to the opposite face of the other, it follows that the vertex of each angle of one triangle is the pole of the opposite side of the other. Hence, such triangles are called *polar* triangles, though sometimes *supplementary.*

776. Theorem.—*If with the several vertices of a spherical triangle as poles, arcs of great circles be made, then a* *second triangle is formed whose vertices are also poles of the first.*

777. Theorem.—*Each angle of a spherical triangle is the supplement of the opposite side of its polar triangle.*

Let ABC be the given triangle, and EF, DF, and DE be arcs of great circles, whose poles are respectively A, B, and C. Then ABC and DEF are polar or supplementary triangles.



These two theorems are corollaries of Article 589, but they can be demonstrated by the student, with the aid of the above diagram, without reference to the triedrals.

778. The student will derive much assistance from drawing the diagrams on a globe. Draw the polar triangle of each of the following: a birectangular triangle, a trirectangular triangle, and a triangle with one side longer than a quadrant and the adjacent angles very acute.

INSCRIBED AND CIRCUMSCRIBED.

779. A sphere is said to be *inscribed* in a polyedron when the faces are tangent to the curved surface, in which case the polyedron is *circumscribed* about the sphere. A sphere is *circumscribed* about a polyedron when the vertices all lie in the curved surface, in which case the polyedron is *inscribed* in the sphere.

780. Problem.—*Any tetraedron may have a sphere inscribed in it; also, one circumscribed about it.*

For within any tetraedron, there is a point equally distant from all the faces (625), which may be the center of the inscribed sphere, the radius being the perpendicular distance from this center to either face. There is also a point equally distant from all the vertices of any tetraedron (623), which may be the center of the circumscribed sphere, the radius being the distance from this point to either vertex.

781. Corollary.—A spherical surface may be made to pass through any four points not in the same plane.

EXERCISES.

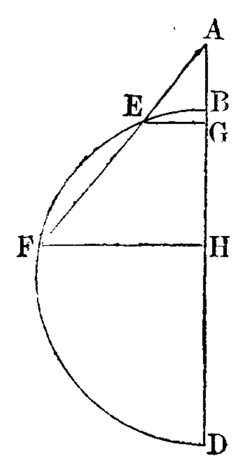
782.—1. In a spherical triangle, the greater side is opposite the greater angle; and conversely.

2. If a plane be tangent to a sphere, at a point on the circumference of a section made by a second plane, then the intersection of these planes is a tangent to that circumference.

3. When two spherical surfaces intersect each other, the line of intersection is a circumference of a circle; and the straight line which joins the centers of the spheres is the axis of that circle.

SPHERICAL AREAS.

783. Let AHF be a right angled triangle and BFD a semicircle, the hypotenuse AF being a secant, and the vertex F in the circumference. From E, the point where AF cuts the arc, let a perpendicular EG fall upon AD.



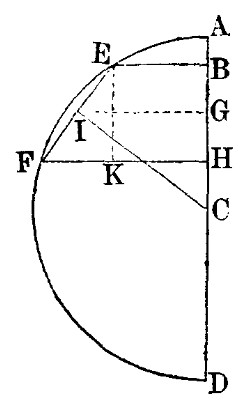
Suppose the whole of this figure to revolve about AD as an axis. The triangle AFH describes a cone, the trapezoid EGHF describes the frustum of a cone, and the semicircle describes a sphere.

The points E and F describe the circumferences of the bases of the frustum; and these circumferences lie in the surface of the sphere.

A frustum of a cone is said to be *inscribed* in a sphere, when the circumferences of its bases lie in the surface of the sphere.

784. Theorem.—*The area of the curved surface of an inscribed frustum of a cone, is equal to the product of the altitude of the frustum by the circumference of a circle whose radius is the perpendicular let fall from the center of the sphere upon the slant hight of the frustum.*

Let AEFD be the semicircle which describes the given sphere, and EBHF the trapezoid which describes the frustum. Let IC be the perpendicular let fall from the center of the sphere upon the slant hight EF.



Then the circumference of a circle of this radius would be π times twice CI, or 2πCI; and it is to be proved that the area of the curved surface of the frustum is equal to the product of BH by 2πCI.

The chord EF is bisected at the point I (187). From this point, let a perpendicular IG fall upon the axis AD. The point I in its revolution describes the circumference of the section midway between the two bases of the frustum. GI is the radius of this circumference, which is therefore 2πGI. The area of the curved surface of the frustum is equal to the product of the slant hight by this circumference (727); that is, EF by 2πGI.

Now from E, let fall the perpendicular EK upon FH. The triangles EFK and IGC, having their sides respectively perpendicular to each other, are similar. Therefore, EF : EK : : CI : GI. Substituting for the second term

EK its equal BH, and for the second ratio its equimultiple $2\pi CI : 2\pi GI$, we have

$$EF : BH :: 2\pi CI : 2\pi GI.$$

By multiplying the means and the extremes,

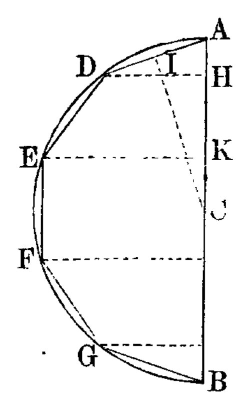
$$EF \times 2\pi IG = BH \times 2\pi IC.$$

But the first member of this equation has been shown to be equal to the area of the curved surface of the frustum. Therefore, the second is equal to the same area.

785. Corollary.—If the vertex of the cone were at the point A, the cone itself would be inscribed in the sphere; and there would be the same similarity of triangles, and the same reasoning as above. It may be shown that the curved surface of an inscribed cone is equal to the product of its altitude by the circumference of a circle whose radius is a perpendicular let fall from the center of the sphere upon the slant hight.

786. Theorem.—*The area of the surface of a sphere is equal to the product of the diameter by the circumference of a great circle.*

Let ADEFGB be the semicircle by which the sphere is described, having inscribed in it half of a regular polygon which may be supposed to revolve with it about the common diameter AB.

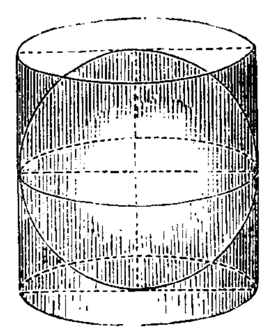


Then, the surface described by the side AD is equal to $2\pi CI$ by AH. The surface described by DE is equal to $2\pi CI$ by HK, for the perpendicular let fall upon DE is equal to CI; and so on. If one of the sides, as EF, is parallel to the axis, the measure is the same, for the surface is cylindrical. Adding these several equations together, we find that the entire surface described by the revolution of the regular polygon about its diameter, is equal to the product of the circumference whose radius is CI, by the diameter AB.

This being true as to the surface described by the perimeter of any regular polygon, it is therefore true of the surface described by the circumference of a circle. But this surface is that of a sphere, and the radius CI then becomes the radius of the sphere. Therefore, the area of the surface of a sphere is equal to the product of the diameter by the circumference of a great circle.

787. Corollary.—The area of the surface of a sphere is four times the area of a great circle. For the area of a circle is equal to the product of its circumference by one-fourth of the diameter.

788. Corollary.—The area of the surface of a sphere is equal to the area of the curved surface of a circumscribing cylinder; that is, a cylinder whose bases are tangent to the surface of the sphere.



AREAS OF ZONES.

789. A Zone is a part of the surface of a sphere included between two parallel planes. That portion of the sphere itself, so inclosed, is called a *segment.* The circular sections are the *bases* of the segment, and the distance between the parallel planes is the *altitude* of the zone or segment.

One of the parallel planes may be a tangent, in which case the segment has one base.

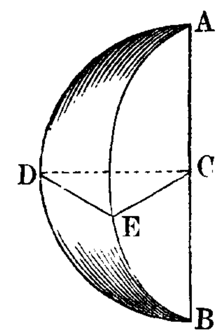
790. Theorem.—*The area of a zone is equal to the product of its altitude by the circumference of a great circle.*

This is a corollary of the last demonstration (786). The area of the zone described by the arc AD, is equal to the product of AH by the circumference whose radius is the radius of the sphere.

AREAS OF LUNES.

791. Theorem.—*The area of a lune is to the area of the whole spherical surface as the angle of the lune is to four right angles.*

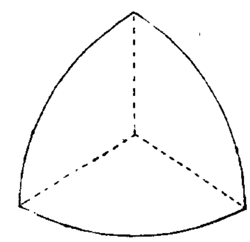
It has already been shown that the angle of the lune is measured by the arc of a great circle whose pole is at the vertex. Thus, if AB is the axis of the arc DE, then DE measures the angle DAE, which is equal to the angle DCE. But evidently the lune varies exactly with the angle DCE or DAE. This may be rigorously demonstrated in the same manner as the principle that angles at the center have the same ratio as their intercepted arcs.



Therefore, the area of the lune has the same ratio to the whole surface as its angle has to the whole of four right angles.

TRIRECTANGULAR TRIANGLE.

792. If the planes of two great circles are perpendicular to each other, they divide the surface into four equal lunes. If a third circle be perpendicular to these two, each of the four lunes is divided into two equal triangles, which have their angles all right angles and their sides all quadrants. Hence, this is sometimes called the *quadrantal* triangle.



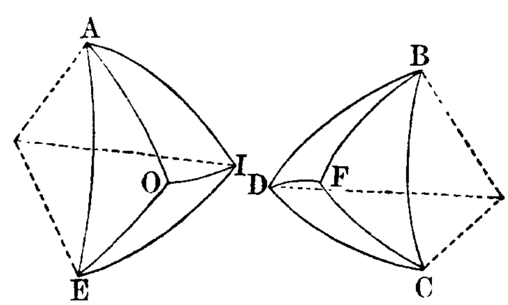
This triangle is the eighth part of the whole surface, as just shown. Its area, therefore, is one-half that of a great circle (787). Since the area of the circle is π times the square of the radius, the area of a trirectangular triangle may be expressed by $\frac{1}{2}\pi R^2$.

The area of the trirectangular triangle is frequently assumed as the unit of spherical areas.

AREAS OF SPHERICAL TRIANGLES.

793. Theorem.—*Two symmetrical spherical triangles are equivalent.*

Let the angle A be equal to B, E to C, and I to D. Then it is known that the other parts of the triangle are respectively equal, but not superposable; and it is to be proved that the triangles are equivalent.



Let a plane pass through the three points A, E, and I; also, one through B, C, and D. The sections thus made are small circles, which are equal; since the distances between the given points are equal chords, and circles described about equal triangles must be equal. Let O be that pole of the first circle which is on the same side of the sphere as the triangle, and F the corresponding pole of the second small circle. Let O be joined by arcs of great circles OA, OE, and OI, to the several vertices of the first triangle; and, in the same way, join FB, FC, and FD.

Now, the triangles AOI and BFD are isosceles, and mutually equilateral; for AO, IO, BF, and DF are equal arcs (753). Hence, these triangles are equal (772). For a similar reason, the triangles IOE and CFD are equal; also, the triangles AOE and BFC. Therefore, the triangles AEI and BCD, being composed of equal parts, are equivalent.

The pole of the small circle may be outside of the given triangle, in which case the demonstration would be by subtracting one of the isosceles triangles from the sum of the other two.

794. It has been shown that the sum of the angles of a spherical triangle is greater than the sum of the angles of a plane triangle (771). Since any spherical polygon can be divided into triangles in the same manner as a plane polygon, it follows that the sum of the angles of any spherical polygon is greater than the sum of the angles of a plane polygon of the same number of sides.

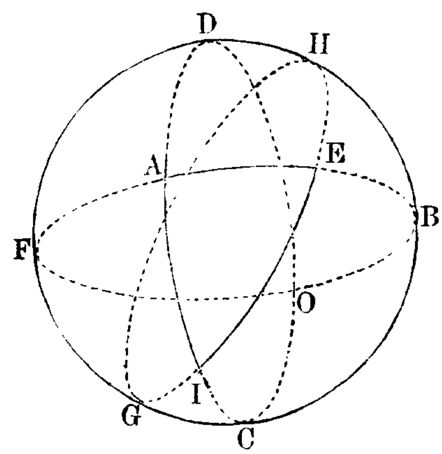
The difference between the sum of the angles of a spherical triangle, or other polygon, and the sum of the angles of a plane polygon of the same number of sides, is called the *spherical excess.*

795. Theorem.—*The area of a spherical triangle is equal to the area of a trirectangular triangle, multiplied by the ratio of the spherical excess of the given triangle to one right angle.*

That is, the area of the given triangle is to that of the trirectangular triangle, as the spherical excess of the given triangle is to one right angle.

Let AEI be any spherical triangle, and let DHBCGF be any great circle, on one side of which is the given triangle. Then, considering this circle as the plane of reference of the figure, produce the sides of the triangle AEI around the sphere.

Now, let the several angles of the given triangle be represented by a, e, and i; that is, taking a right angle for the unit, the angle EAI is equal to a right angles, etc. Then, the area of the lune AEBOCI is to the whole surface as a is to 4 (791). But if the trirectangular triangle, which is one-eighth of the spherical surface, be taken as the unit of area, then the area of this lune is $2a$. But the triangle BOC, which is a part of this lune, is equivalent to its opposite and symmetrical triangle DAF. Substituting this latter, the area of the two triangles ABC and DAF is $2a$ times the unit of area.



In the same way, show that the area of the two triangles IDH and IGC is $2i$, and that the area of the two triangles EFG and EHB is $2e$ times the unit of area. These equations may be written thus:

area $(\text{ABC} + \text{ADF}) = 2a$ times the trirectangular triangle;

area $(\text{IDH} + \text{IGC}) = 2i$ times the trirectangular triangle;

area $(\text{EFG} + \text{EHB}) = 2e$ times the trirectangular triangle.

In adding these equations together, take notice that the triangles mentioned include the given triangle AEI three times, and all the rest of the surface of the hemisphere above the plane of reference once; also, that the area of this hemispherical surface is four times that of the trirectangular triangle. Then, by addition of the equations:

area 4 trirect. tri. + 2 area AEI = 2 ($a + e + i$) trir. tri.

Transposing the first term, and dividing by 2,

area AEI = ($a + e + i - 2$) trir. tri.

But ($a + e + i - 2$) is the spherical excess of the given triangle, taking a right angle as a unit; that is, it is the ratio of the spherical excess of the given triangle to one right angle.

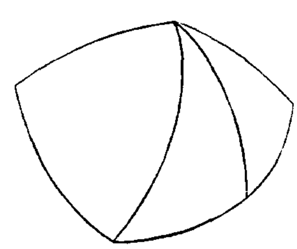
796. Corollary.—If the square of the radius be taken as the unit of area, then the area of any spherical triangle may be expressed (792),

$$\tfrac{1}{2}(a + e + i - 2)\pi R^2.$$

AREAS OF SPHERICAL POLYGONS.

797. Theorem.—*The area of any spherical polygon is equal to the area of the trirectangular triangle multiplied by the ratio of the spherical excess of the polygon to one right angle.*

For the spherical excess of the polygon is evidently the sum of the spherical excess of the triangles which compose it; and its area is the sum of their areas.



EXERCISES.

798.—1. What is the area of the earth's surface, supposing it to be in the shape of a sphere, with a diameter of 7912 miles?

2. Upon the same hypothesis, what portion of the whole surface is between the equator and the parallel of 30° north latitude?

3. Upon the same hypothesis, what portion of the whole surface is between two meridians which are ten degrees apart?

4. What is the area of a triangle described on a globe of 13 inches diameter, the angles being 100°, 45°, and 53°?

VOLUME OF THE SPHERE.

799. Theorem.—*The volume of any polyedron in which a sphere can be inscribed, is equal to one-third of the product of the entire surface of the polyedron by the radius of the inscribed sphere.*

For, if a plane pass through each edge of the polyedron, and extend to the center of the sphere, these planes will divide the polyedron into as many pyramids as the figure has faces. The faces of the polyedron are the bases of the pyramids.

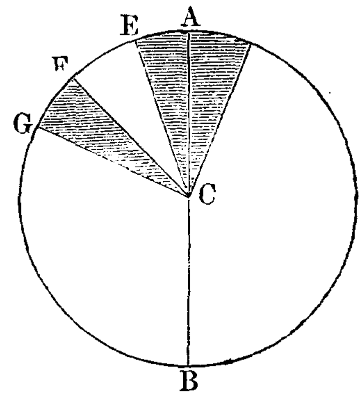
The altitude of each is the radius of the sphere, for the radius which extends to the point of tangency is perpendicular to the tangent plane (742). But the volume of each pyramid is one-third of its base by its altitude. Therefore, the volume of the whole polyedron is one-third the sum of the bases by the common altitude, or radius.

800. Theorem.—*The volume of a sphere is equal to one-third of the product of the surface by the radius.*

For, the surface of a sphere may be approached as nearly as we choose, by increasing the number of faces of the circumscribing polyedron, until it is evident that the sphere is the limit of the polyedrons in which it is inscribed. Then, this theorem becomes merely a corollary of the preceding.

801. Corollary.—The volume of a spherical pyramid, or of a spherical wedge, is equal to one-third of the product of its spherical surface by the radius.

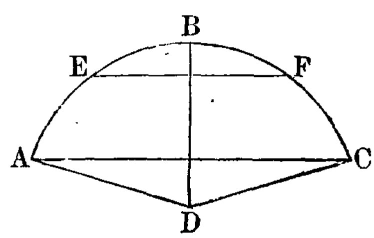
802. A spherical SECTOR is that portion of a sphere which is described by the revolution of a circular sector about a diameter of the circle. It may have two or three curved surfaces.



Thus, if AB is the axis, and the generating sector is AEC, the sector has one spherical and one conical surface; but if, with the same axis, the generating sector is FCG, then the sector has one spherical and two conical surfaces.

803. Corollary.—The volume of a spherical sector is equal to one-third of the product of its spherical surface by the radius.

804. The volume of a spherical segment of one base is found by subtracting the volume of a cone from that of a sector. For the sector ABCD is composed of the segment ABC and the cone ACD.



The volume of a spherical segment of two bases is the difference of the volumes of two segments each of one base. Thus the segment AEFC is equal to the segment ABC less EBF.

805. All spheres are similar, since they are generated by circles which are similar figures. Hence, we might at once infer that their surfaces, as well as their volumes, have the same ratios as in other similar solids. These principles may be demonstrated as follows:

806. Theorem. — *The areas of the surfaces of two spheres are to each other as the squares of their diameters; and their volumes are as the cubes of their diameters, or other homologous lines.*

For the superficial area of any sphere is equal to π times the diameter multiplied by the diameter (786); that is πD^2. But π is a certain or constant factor. Therefore, the areas vary as the squares of the diameters.

The volume is equal to the product of the surface by one-sixth of the diameter (800); that is, πD^2 by $\frac{1}{6}D$, or $\frac{1}{6}\pi D^3$. But $\frac{1}{6}\pi$ is a constant numeral. Therefore, the volumes vary as the cubes of the diameters.

USEFUL FORMULAS.

807. Represent the radius of a circle or a sphere, or that of the base of a cone or cylinder, by R; represent the diameter by D, the altitude by A, and the slant hight by H.

Circumference of a circle	$=\pi D=2\pi R$,
Area of a circle	$=\frac{1}{4}\pi D^2=\pi R^2$,
Curved surface of a cone	$=\frac{1}{2}\pi DH=\pi RH$,
Entire surface of a cone	$=\pi R(H+R)$,
Volume of a cone	$=\frac{1}{12}\pi D^2A=\frac{1}{3}\pi R^2A$,
Curved surface of a cylinder	$=\pi DA=2\pi RA$,
Entire surface of a cylinder	$=2\pi R(A+R)$,
Volume of a cylinder	$=\frac{1}{4}\pi D^2A=\pi R^2A$,
Surface of a sphere	$=\pi D^2=4\pi R^2$,
Volume of a sphere	$=\frac{1}{6}\pi D^3=\frac{4}{3}\pi R^3$,

$$\pi=3.1415926535.$$

EXERCISES.

808.—1. What is the locus of those points in space which are at the same distance from a given point?

2. What is the locus of those points in space which are at the same distance from a given straight line?

3. What is the locus of those points in space, such that the distance of each from a given straight line, has a constant ratio to its distance from a given point of that line?

EXERCISES FOR GENERAL REVIEW.

809.—1. Take some principle of general application, and state all its consequences which are found in the chapter under review; arranging as the first class those which are immediately deduced from the given principle; then, those which are derived from these, and so on.

2. Reversing the above operation, take some theorem in the latter part of a chapter, state all the principles upon which its proof immediately depends; then, all upon which these depend; and so on, back to the elements of the science.

3. Given the proportion, $a : b :: c : d$,
to show that $c-a : d-b :: a : b$;
also, that $a+c : a-c :: b+d : b-d$.

4. Form other proportions by combining the same terms.

5. What is the greatest number of points in which seven straight lines can cut each other, three of them being parallel; and what is the least number, all the lines being in one plane?

6. If two opposite sides of a parallelogram be bisected, straight lines from the points of bisection to the opposite vertices will trisect the diagonal.

7. In any triangle ABC, if BE and CF be perpendiculars to any line through A, and if D be the middle of BC, then DE is equal to DF.

8. If, from the vertex of the right angle of a triangle, there extend two lines, one bisecting the base, and the other perpen-

dicular to it, the angle of these two lines is equal to the difference of the two acute angles of the triangle.

9. In the base of a triangle, find the point from which lines extending to the sides, and parallel to them, will be equal.

10. To construct a square, having a given diagonal.

11. Two triangles having an angle in the one equal to an angle in the other, have their areas in the ratio of the products of the sides including the equal angles.

12. If, of the four triangles into which the diagonals divide a quadrilateral, two opposite ones are equivalent, the quadrilateral has two opposite sides parallel.

13. Two quadrilaterals are equivalent when their diagonals are respectively equal, and form equal angles.

14. Lines joining the middle points of the opposite sides of any quadrilateral, bisect each other.

15. Is there a point in every triangle, such that any straight line through it divides the triangle into equivalent parts?

16. To construct a parallelogram having the diagonals and one side given.

17. The diagonal and side of a square have no common measure, nor common multiple. Demonstrate this, without using the algebraic theory of radical numbers.

18. To construct a triangle when the three altitudes are given.

19. To construct a triangle, when the altitude, the line bisecting the vertical angle, and the line from the vertex to the middle of the base, are given.

20. If from the three vertices of any triangle, straight lines be extended to the points where the inscribed circle touches the sides, these lines cut each other in one point.

21. What is the area of the sector whose arc is 50°, and whose radius is 10 inches?

22. To construct a square equivalent to the sum, or to the difference of two given squares.

23. To divide a given straight line in the ratio of the areas of two given squares.

24. If all the sides of a polygon except one be given, its area will be greatest when the excepted side is made the diameter of a circle which circumscribes the polygon.

25. Find the locus of those points in a plane, such that the sum of the squares of the distances of each from two given points, shall be equivalent to the square of a given line.

26. Find the locus of those points in a plane, such that the difference of the squares of the distances of each from two given points, shall be equivalent to the square of a given line.

27. If the triangle DEF be inscribed in the triangle ABC, the circumferences of the circles circumscribed about the three triangles AEF, BFD, CDE, will pass through the same point.

28. The three points of meeting mentioned in Exercises 28, 29, and 30, Article 337, are in the same straight line.

29. If, on the sides of a given plane triangle, equilateral triangles be constructed, the triangle formed by joining the centers of these three triangles is also equilateral; and the lines joining their vertices to the opposite vertices of the given triangle are equal, and intersect in one point.

30. The feet of the three altitudes of a triangle and the centers of the three sides, all lie in one circumference. The circle thus described is known as "The Six Points Circle."

31. Four circles being described, each of which shall touch the three sides of a triangle, or those sides produced; if six lines be made, joining the centers of those circles, two and two, then the middle points of these six lines are in the circumference of the circle circumscribing the given triangle.

32. If two lines, one being in each of two intersecting planes, are parallel to each other, then both are parallel to the intersection of the planes.

33. If a line is perpendicular to one of two perpendicular planes, it is parallel to the other; and, conversely, if a line is parallel to one and perpendicular to another of two planes, then the planes are perpendicular to each other.

34. How may a pyramid be cut by a plane parallel to the base, so as to make the area or the volume of the part cut off have a given ratio to the area or the volume of the whole pyramid?

35. Any regular polyedron may have a sphere inscribed in it; also, one circumscribed about it.

36. In any polyedron, the sum of the number of vertices and the number of faces exceeds by two the number of edges.

37. How many spheres can be made tangent to three given planes?

38. A frustum of a pyramid is equivalent to the sum of three pyramids having the same altitude as the frustum, and having for their bases the lower base of the frustum, the upper base, and a mean proportional between them.

39. The surface of a sphere can be completely covered with the surfaces either of 4, or of 8, or of 20 equilateral spherical triangles.

40. The volume of a cone is equal to the product of its whole surface by one-third the radius of the inscribed sphere.

41. If, about a sphere, a cylinder be circumscribed, also a cone whose slant height is equal to the diameter of its base, then the area and volume of the sphere are two-thirds of the area and volume of the cylinder; and the area and volume of the cylinder are two-thirds of the area and volume of the cone.

MENSURATION.

810. MENSURATION, or the art of measuring, consists in rules for the measurement of lines, surfaces, and solids.

Exercises have been given in the previous pages requiring the application of the principles of Geometry to various kinds of measurement (393, 414, 472, 506, 507, 668, 685, 706, 715, 798, and 809). The formulas of Art. 807 also afford many useful rules of Mensuration, each of which should be applied by the student to particular examples.

A few trigonometrical principles will, with the exercises given hereafter, complete all that is usually included in this branch of applied mathematics.

TRIGONOMETRY.

CHAPTER XII.

PLANE TRIGONOMETRY.

811. TRIGONOMETRY is the science in which the relations subsisting between the angles, sides, and area of any triangle are investigated. The science was originally called *Plane* Trigonometry or *Spherical* Trigonometry, according as the triangle was plane or spherical.

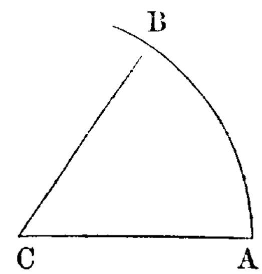
PLANE TRIGONOMETRY has now a wider meaning, comprising algebraic investigations concerning angles and their functions, and the methods of calculating these functions.

MEASURE OF ANGLES.

812. In Elementary Geometry, the unit for the measure of angles is usually the right angle. The frequent fractions which the use of this unit gives rise to, render it inconvenient for calculation. It has been divided into degrees, minutes, and seconds (208).

This sexagesimal division of angles has been in use since the second century. Efforts have been made to substitute for it the centesimal division, making the right angle contain one hundred *grades*, each grade one hundred *minutes*, and so on; but this plan has never been generally in use.

813. There is another unit which has been called the *circular measure* of an angle. It is used in trigonometrical investigation, and is also called the *analytical* unit. It is that angle at the center of a circle whose intercepted arc has the same linear extent as the radius. Thus, if the arc AB has the same linear extent as the radius AC, then the angle C is the unit of circular measure. Hence, this unit of measure is equal to

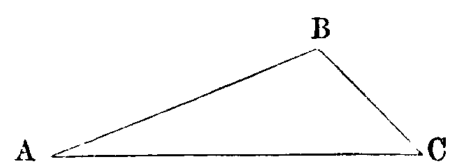


$$\frac{180^\circ}{\pi}=57^\circ.\ 29578—=57^\circ\ 17'\ 44''.\ 8+.$$

Also, $1^\circ=\frac{\pi}{180}=.017453$ times the circular measure.

814. Various instruments are used for the measure of angles. A *protractor* is used to measure the angle of two lines in a drawing. It is usually shaped like a semicircumference with its diameter, the arc being marked with the degrees from 0 to 180.

Let it be required to measure the angle ABC. Place the center of the straight edge, which is marked by a notch on the instrument, at the vertex B; let the edge lie along one side of the angle, as BC; then read the degree marked where the other side BA passes the arc of the instrument. This gives the size of the angle.



The same instrument is used for drawing angles of a known size. One side of the angle being drawn, place the center of the protractor at the point which is to be the vertex; then the required number of degrees, on the edge of the arc, will indicate a point on the other side of the angle. Connect this point with the vertex to complete the angle.

The student should be provided with a protractor, a six-inch scale, and a pair of dividers. Large protractors, made of wood, pasteboard, or tin-plate, are useful for blackboard work.

EXERCISES.

315.—1. Find the circular measure of an angle of 3° 4′ 5″.

2. Draw a triangle having one side two inches, another three inches, and the included angle 100°. Find the other angles and side by measurement.

3. Draw a triangle with the sides three, four, and five inches in length. Find the angles by measurement.

These exercises may be extended and varied, referring to Articles 295 to 301 inclusive.

FUNCTIONS OF ANGLES.

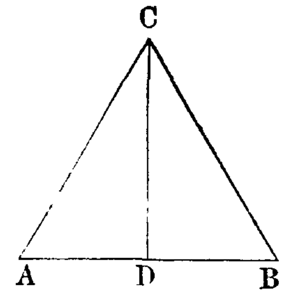
316. When two quantities are so related that any variation in one causes a variation in the other, each is a *function* of the other. Thus, $\sqrt{x}$ is a function of x; the area of a circle is a function of its radius (500).

A quantity may be a function of several others. Thus, $x^2 y^3$ is a function of x and y; the area of a triangle is a function of its base and altitude (386). The angles of a triangle are functions of the ratios of the sides (316); and the ratios of the sides of a triangle are functions of the angles (309).

For example, if the lengths of the sides be as the numbers 3, 4, and 5, then the angle opposite the longest side is a right angle (413); and each of the acute angles is also a function of the numbers 3, 4, and 5.

For another example; if the triangle ACD has its angles 30°, 60°, and 90°, then it may be shown that

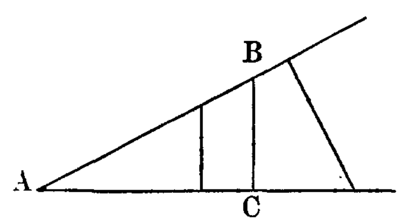
$$\frac{AC}{AD}=2,\ \frac{CD}{AD}=\sqrt{3},\ \text{and}\ \frac{AC}{CD}=\tfrac{2}{3}\sqrt{3}.$$



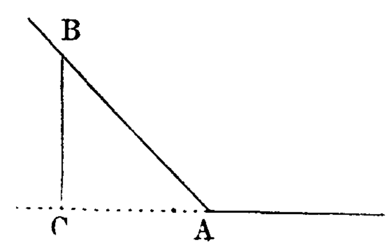
Let the student now solve the 1st Exercise of Art. 472.

817. Theorem.—*If from any point in one side of an angle, a perpendicular fall upon the other side, a right angled triangle is formed, and the ratios of the sides of this triangle are functions of the given angle.*

For, if any number of triangles were thus formed with a given angle, all of these triangles would be similar (306), and their sides would have the same ratios (309).



When the given angle is greater than a right angle, one side may be produced to meet the perpendicular.



818. If from any point on one side of a given angle a perpendicular fall on the other side as a base, then

The Sine of the given angle is the ratio of the perpendicular to the hypotenuse of the right angled triangle thus formed.

The Tangent of the angle is the ratio of the perpendicular to the base.

The Secant of the angle is the ratio of the hypotenuse to the base.

The Cosine of the angle is the ratio of the base to the hypotenuse.

The COTANGENT of the angle is the ratio of the base to the perpendicular.

The COSECANT of the angle is the ratio of the hypotenuse to the perpendicular.

The abbreviations sin., tan., sec., cos., cot., and cosec. are used respectively for these six functions. Thus, the sine of the angle A is written sin. A.

These six are all the ratios that can be formed by the simple combination of the sides of the triangle. They are called, therefore, the simple functions of an angle. Other functions have been formed by composition and by division. Of these, the following is used at the present day:

The VERSED SINE is the ratio of the excess of the hypotenuse over the base, to the hypotenuse. Hence,

$$\text{vers. sin. } A = 1 - \text{cos. } A.$$

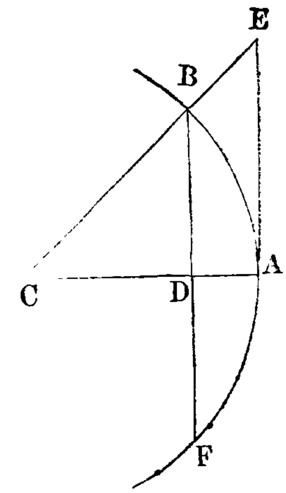
819. A table of sines of every degree from 0 to 90° may be made by drawing and measurement. Draw a right angled triangle, with an angle at the base equal to the angle whose sine we wish to find. It will simplify the work to make the hypotenuse the length of a certain unit. Divide the length of the perpendicular by that of the hypotenuse. The quotient is the sine. By careful drawing and measurement, a table of sines may be made that shall be true to two places of decimals.

A table of tangents may be formed in a similar manner, making the base the length of a certain unit.

By calculation, the functions may all be found to any required degree of accuracy.

820. The etymology of sine, tangent, and secant appears from the method which was formerly used to define these terms, which was as follows:

If with any radius an arc be described about the vertex C as a center, and if from B, one extremity of the intercepted arc, a perpendicular BD fall upon the side CA, then BD is called the sine of the arc BA, or of the angle C. If a perpendicular to AC be produced to meet CB at E, then AE is the tangent and CE the secant of the arc AB, or of the angle C.



The student readily perceives that if the radius is taken as the unit of length, then the lengths of BD, AE, and CE are respectively the sine, tangent, and secant of the angle C. The names tangent and secant are taken from the geometrical tangent and secant. Arc, chord, and sine are derived from the fancied resemblance of the figure to the bow of the archer. The curve BAF is the bow or *arc*, the *chord* BF joins its ends, and BD touches the breast or *sinus* of the archer. So also DA has been called the *sagitta* or arrow. When used now, it is called the versed sine.

The oldest work on Trigonometry now extant is the Almagest of Ptolemy, written in the second century. He divides the radius into sixty parts, also the arc whose chord is equal to the radius into the same number of parts. This mode of measuring arcs, and consequently angles, remains in use, but the sexagesimal division of lines was long since abandoned. The use of sines was introduced by the Arabian mathematicians about the eighth or ninth century. Napier, a Scotch Baron, who lived in the early part of the seventeenth century, has done more for the science of Trigonometry than any other one man.

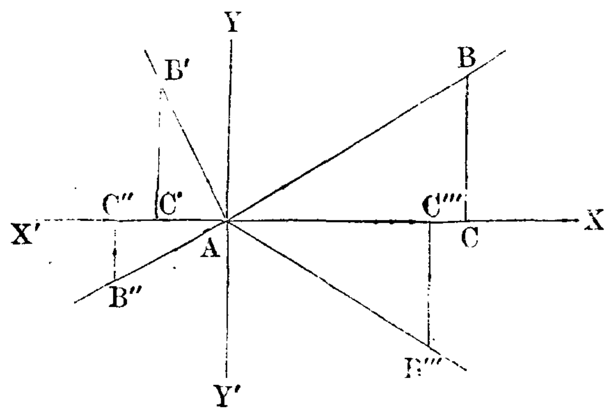
EXERCISES.

821.—1. Demonstrate tan. $45°=1$; also, sin. $60°=\frac{1}{2}\sqrt{3}$.

2. Construct an angle whose sine is $\frac{2}{3}$; one whose tangent is $\frac{4}{5}$; one whose secant is $\frac{7}{4}$.

SIGNS OF ANGLES AND OF THEIR FUNCTIONS.

822. An angle may be conceived to be generated by the revolution of a line about a point. Thus, the line AB beginning at AX, may take the positions AB, AB′, AB″, AB‴, AX, and so on indefinitely, repeating at each revolution all the positions of the first.



In Trigonometry, the amount of this revolution is considered as an angle, so that an angle may be greater than the sum of two or of four right angles. In the strict geometrical definition, an angle being the difference of two directions, can not be greater than two right angles.

Quantities conceived to exist in a certain direction or mode are called positive, and are designated by the sign +; while the quantities in the opposite direction are called negative, and are designated by the sign —.

In the present investigation, the angle is supposed to be generated by the motion of the line AB *up* from AX. Angles so formed are positive, and when estimated in the opposite direction they are negative. Thus, if BAC is an acute angle, it is positive. If it is negative, it is greater than the sum of three right angles. The complement of an angle greater than a right angle must be negative, and the same is true of the supplement of an angle greater than two right angles.

The directions to the right of YY′ and those upwards from XX′ are positive. Then the directions to the left from YY′ and those downward from XX′ are negative. Thus, AC, CB, and C′B′ are positive, while AC′, C″B″, and C‴B‴ are negative.

823. Theorem.—*The functions of any acute angle are positive.*

For when the revolving line is in the first quarter of its revolution, that is, between AX and AY, all the sides of the triangle ABC are positive.

The same is true of the functions of any angle which is equal to $4n$ right angles $+$ an acute angle, n being any entire number positive or negative.

824. Theorem.—*The tangent, secant, cosine, and co-tangent of obtuse angles are negative, while the sine and cosecant of obtuse angles are positive.*

For, when the revolving line is in the second quarter of its revolution, that is, between AY and AX′, the side AC′ of the triangle AB′C′ is the only negative term. Hence, the functions of which it forms one term are negative.

The same is true of the respective functions of any angle which is equal to an obtuse angle $\pm$ $4n$ right angles.

825. In this manner, the signs of the functions may be found, and arranged according to the quarter of the revolving line AB. The following table exhibits the signs of the functions of all angles whatsoever:

REVOLVING LINE IN	SINE & COSEC.	COS. & SEC.	TAN. & COT.
First quarter, ...	+	+	+
Second quarter, . . .	+	—	—
Third quarter, ...	—	—	+
Fourth quarter, . . .	—	+	—

826. Corollary.—The functions of two angles are the same, when one of the angles is greater than the other by four right angles.

EXERCISES.

827.—1. Demonstrate the following equations: sec. 120°$=-2$; cos. 135°$=-\frac{1}{2}\sqrt{2}$.

2. The ratio of one straight line to its projection upon another is what function of their angle?

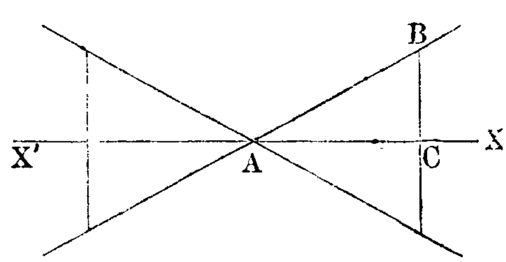
3. Construct an angle whose tangent is -1; one whose sine is $-\frac{1}{2}$.

4. Construct an angle whose cosine is $-\frac{2}{3}$.

ANGLES OF A GIVEN FUNCTION.

828. Theorem.—*Any given simple function, when taken irrespective of its algebraic sign, belongs to four different angles within each revolution.*

If BAC is the acute angle of a given function, the revolving line AB will, at some point in each quarter of its revolution, form an acute angle with XX′, equal to the angle BAC. Now, the numerical value of the function



depends upon the acute angle which the revolving line makes with the fixed line (817). Hence, there is an angle for each quarter whose functions are numerically equal to those of the angle BAC.

829. Corollary.—Any simple function of an angle is numerically equal to the same function of

1st. The supplement of the angle;
2nd. The given angle increased by two right angles;
3rd. The given angle taken negatively.

The sine and cosecant of supplementary angles have the same signs, while the other simple functions of supplementary angles have opposite signs (825). The cosine and secant of an angle and of its negative have the same signs, while the other simple functions of such angles have opposite signs. The tangent and cotangent of an angle, and of the same angle increased by two right angles, have the same signs, while the other simple functions of such angles have opposite signs.

These conclusions as to the sine may be expressed thus:

$$\sin. A = \sin. (180° - A) = -\sin. (180° + A) = -\sin. (-A).$$

The following more general expressions are easily deduced from the above corollary. If n is 0, or any integer positive or negative, and A is any angle, then

The formula $n \cdot 180° + (-1)^n A$ includes all angles which have the same sine as A;

The formula $n \cdot 360° \pm A$ includes all the angles which have the same cosine as A; and

The formula $n \cdot 180° + A$ includes all angles which have the same tangent as A.

830. Corollary.—Any simple function of any angle may be expressed in terms of the same function of an acute angle.

EXERCISES.

831.—1. Make a formula analogous to the above for each of the other simple functions.

2. Demonstrate cosec. $600° = -\frac{2}{3}\sqrt{3}$; cot. $405° = 1$.

3. Write a formula containing all the values of A when tan. A = 1.

LIMITS OF FUNCTIONS.

832. Theorem.—*The sine of any angle can not be greater than* 1, *nor less than* -1; *and the cosine has the same limits.*

For the leg of a right angled triangle can not be greater than the hypotenuse; and, therefore, the sine and cosine are fractions having the numerator less than the denominator.

833. Theorem.—*The secant and cosecant can not have any values between* 1 *and* -1; *and the tangent and cotangent have no limits.*

These principles also follow immediately from the definitions and the nature of a right angled triangle.

834. As the revolving line passes through the first quarter of its revolution, the sine increases from 0 to 1. The sine of a right angle is unity, for in that case the perpendicular coincides with the hypotenuse. Then the sine decreases till the angle is equal to two right angles, when the sine becomes 0. It continues to decrease till the angle becomes three right angles, when the sine is -1. Then again it increases to the end of the revolution, where the sine is 0.

The cosine of 0° is 1, which decreases as the angle increases till the cosine of 90° is 0, and the cosine of

180° is −1. Then it increases through the remaining half of the revolution.

The tangent of 0° is 0. As the angle increases the tangent increases without limit, and the tangent of a right angle is infinite. The tangent of an obtuse angle is negative, and as the angle increases the tangent varies from minus infinity to zero. In the third quarter the tangent varies as in the first quarter through all possible positive values; and the variations of the fourth quarter are like those of the second.

The variations of the cotangent, secant, and cosecant may be traced in the same way.

These values of the functions at particular points may be expressed as follows:

	SIN.	COS.	TAN.	COT.	SEC.	COSEC.
0°	0	1	0	∞	1	∞
90°	1	0	∞	0	∞	1
180°	0	−1	0	∞	−1	∞
270°	−1	0	∞	0	∞	−1
360°	0	1	0	∞	1	∞

The versed sine increases from 0 to 2 as the angle increases from 0° to 180°, and decreases from 2 to 0 through the other two quarters.

EXERCISES.

335.—1. Trace the value of this expression: cos. A — sin. A, as A varies from 0° to 360°.

2. What are the sine and the tangent of 810°?

3. What are the cosine and secant of —450°?

4. What are the cosecant and cotangent of 150°?

5. Construct an angle greater than 90°, whose sine is $\frac{1}{2}$; one whose tangent is $\frac{1}{2}$; one whose cosine is $\frac{1}{2}$.

RELATIONS BETWEEN THE FUNCTIONS.

836. A simple function of an angle, being a ratio, may be expressed as a fraction.

Let a be the perpendicular, b the base, and c the hypotenuse of the triangle used in defining the functions of an angle. In order to include all possible angles, let it be understood that a and b are either positive or negative. Then,

$$\sin. A = \frac{a}{c}, \qquad \cos. A = \frac{b}{c},$$

$$\tan. A = \frac{a}{b}, \qquad \cot. A = \frac{b}{a},$$

$$\sec. A = \frac{c}{b}, \qquad \text{cosec. } A = \frac{c}{a}.$$

837. Corollary.—The sine and cosecant of an angle are reciprocals; also, the tangent and cotangent are reciprocals; and the cosine and secant are reciprocals. That is,

$$\sin. A \text{ cosec. } A = 1, \quad \tan. A \cot. A = 1, \quad \cos. A \sec. A = 1.$$

A practical result of these equations is, that the cosecant, secant, and cotangent are less used than the other simple functions. For, if one has occasion to multiply or divide by the cosecant, the object is accomplished by dividing or multiplying by the sine; and similarly of the secant and cotangent.

838. By means of the Pythagorean Theorem and the fractions just stated, any function of an angle may be expressed in terms of any other function of the same angle. For example, let it be required to find the value of each of the other simple functions in terms of the sine of the same angle. Beginning with the equation,

$$a^2 + b^2 = c^2,$$

and dividing both members by c^2,

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1.$$

That is, the sum of the squares of the sine and cosine of any angle is equal to unity. Hence,

$$\text{sin. A} = \sqrt{1 - \text{cos.}^2 \text{A}}; \quad \text{also, cos. A} = \sqrt{1 - \text{sin.}^2 \text{A}}.$$

The exponent is given to sin. and to cos., because it is the function that is involved and not the angle.

839. The sine of an angle is equal to the product of the tangent by the cosine. For,

$$\frac{a}{c} = \frac{a}{b} \times \frac{b}{c}.$$

That is, $\text{sin. A} = \text{tan. A cos. A}.$

Hence, $$\text{tan. A} = \frac{\text{sin. A}}{\text{cos. A}} = \frac{\text{sin. A}}{\sqrt{1 - \text{sin.}^2 \text{A}}}.$$

Since the tangent and cotangent are reciprocals,

$$\text{cot. A} = \frac{\text{cos. A}}{\text{sin. A}} = \frac{\sqrt{1 - \text{sin.}^2 \text{A}}}{\text{sin. A}}.$$

Since the secant and cosine are reciprocals,

$$\text{sin. A} = \sqrt{1 - \frac{1}{\text{sec.}^2 \text{A}}} = \frac{\sqrt{\text{sec.}^2 \text{A} - 1}}{\text{sec. A}}.$$

EXERCISES.

840.—1. By similar methods, find expressions for the cosine and tangent in terms of each of the other functions.

2. Render each formula into ordinary language. This valuable exercise should be continued throughout the work.

3. Given 2 sin. A = tan. A, to find A. Ans. 0°, 60°, 120°, 180°, 240°, or 300°.

4. If sin. A = $\frac{3}{5}$, what is the value of cos. A?

5. If sin. A = $\frac{1}{3}$, what is the value of tan. A?

6. Demonstrate sin. 18° = $\frac{1}{4}(\sqrt{5} - 1)$. Notice that 18° is the angle made by the apothegm and radius of a regular decagon.

FUNCTIONS OF (90° ± A).

841. Theorem.—*The cosine of an angle is the sine of its complement.*

That is, cos. A = sin. (90° — A). For, in the right angled triangle of the definitions, the acute angles are complementary; and (818)

$$\cos. A = \frac{b}{c} = \sin. B.$$

This demonstration appears to apply only to the case when the angle A is acute, when the revolving line is in the first quarter. The student may construct a figure for each of the other quarters, and show that the proposition is universally true.

842. Corollary.—Similarly, the cotangent and cosecant are respectively the tangent and secant of the complementary angle. It is from this property that these functions (cos., cot., cosec.) derive their names.

843. Theorem.—*Sin.* $(90° + A) =$ *cos. A, and cos.* $(90° + A) = -$ *sin. A.*

It has been proved that sin. $A =$ sin. $(180° - A)$, whatever is the value of A (829). It is therefore true for $(90° + A)$. Substituting, we have

$$\text{sin.}\,(90°+A) = \text{sin.}\,(180° - 90° - A) = \text{sin.}\,(90° - A) = \text{cos.}\,A.$$

Again, since cos. $A =$ sin. $(90° - A)$ for all values of A, then for A we may substitute $90° + A$. Hence,

$$\text{cos.}\,(90°+A) = \text{sin.}\,(90° - 90° - A) = \text{sin.}\,(-A) = -\text{sin.}\,A.$$

EXERCISES.

844.—1. Find the value of tan. $(90° + A)$.

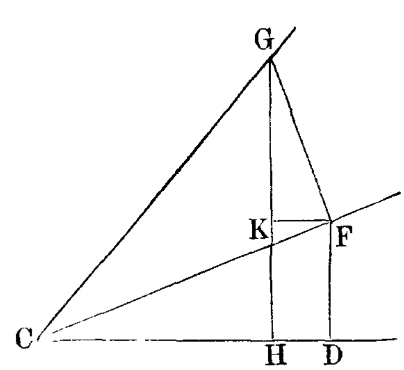
2. Illustrate with diagrams all the principles of this section.

3. Given sin. $A =$ cos. $2A$, to find the value of A.

4. Demonstrate tan. $72° = \dfrac{\sqrt{(10 + 2\sqrt{5})}}{\sqrt{5} - 1}.$

FUNCTIONS OF TWO ANGLES.

845. Let the angle DCF be designated by A and the angle FCG by B; then DCG is A+B. From any point G in the line CG let fall GH and GF respectively perpendicular to CD and CF. From F let fall FD and FK respectively perpendicular to CD and GH. Then, the angle FGK is equal to FCD, or A (140).



Now $DF = CF \times \text{sin.}\,A$, and $CF = CG \times \text{cos.}\,B$; hence,

$$DF = CG \times \text{sin.}\,A\ \text{cos.}\,B.$$

Likewise $GK = GF \times \cos. A$, and $GF = CG \times \sin. B$; hence,

$$GK = CG \times \cos. A \sin. B.$$

Also, $GK + DF = GK + KH = GH = CG \times \sin. (A+B)$; therefore,

$$\sin. (A+B) = \sin. A \cos. B + \cos. A \sin. B, \quad \text{(I.)}$$

In the above figure the given angles and their sum are acute. The same demonstration will apply for any given angles, constructing the figure exactly according to the directions, producing when necessary the lines on which the perpendiculars fall.

The cosine of the sum of two angles may be found in terms of the sine and cosine of the angles, by the above diagram and similar reasoning. Or, it may be derived from the formula just demonstrated, as follows:

Regarding $90° + A$ as one angle, we have

$$\sin. (90°+A+B) = \sin. (90°+A) \cos. B + \cos. (90°+A) \sin. B.$$

Substituting for the functions of $90°+A$ and $90°+A+B$, their equivalents (843),

$$\cos. (A+B) = \cos. A \cos. B - \sin. A \sin. B, \quad \text{(II.)}$$

In these two formulas for the sine and cosine of the sum of two angles, if $-B$ is substituted for B, then the sign of sin. B is changed, but not of cos. B (825). Thus,

$$\sin. (A-B) = \sin. A \cos. B - \cos. A \sin. B, \quad \text{(III.)}$$
$$\cos. (A-B) = \cos. A \cos. B + \sin. A \sin. B, \quad \text{(IV.)}$$

These two formulas may be demonstrated independently of the former, in the same manner as the formula for the sine of the sum.

The tangent of the sum of two angles is found thus:

$$\tan.(A+B)=\frac{\sin.(A+B)}{\cos.(A+B)}=\frac{\sin. A \cos. B+\cos. A \sin. B}{\cos. A \cos. B-\sin. A \sin. B}.$$

Dividing both terms of the fraction by cos. A cos. B,

$$\tan.(A+B)=\frac{\tan. A+\tan. B}{1-\tan. A \tan. B}, \quad . \quad (\text{V.})$$

Similarly, $$\tan.(A-B)=\frac{\tan. A-\tan. B}{1+\tan. A \tan. B}, \quad . \quad (\text{VI.})$$

EXERCISES.

846.—1. Demonstrate formula II in the same manner as formula I, and both of them for those cases where the angles are not acute. Observe in what quarters the sine and cosine are negative.

2. Express each formula in ordinary language; for example: the sine of the sum of two angles is equal to the sum of the products of the sine of each by the cosine of the other.

3. Demonstrate $\cos. 12° = \frac{1}{8}(\sqrt{30+6\sqrt{5}}+\sqrt{5}-1.)$

FUNCTIONS OF MULTIPLES AND PARTS OF ANGLES.

847. In the formulas of the sine, the cosine, and the tangent of the sum of two angles, suppose B = A; then,

$$\sin. 2A = 2 \sin. A \cos. A, \quad . \quad . \quad . \quad . \quad (\text{I.})$$

$$\cos. 2A = \cos.^2 A - \sin.^2 A, \quad . \quad . \quad . \quad (\text{II.})$$

$$\tan. 2A = \frac{2 \tan. A}{1-\tan.^2 A}, \quad . \quad . \quad . \quad . \quad (\text{III.})$$

By substituting $(n-1)A$ for B in the original formulas, sin. nA, cos. nA, and tan. nA may be expressed in functions of A and of $(n-1)A$. Thus, when the functions of

A are known, the functions of 2A, 3A, etc., may be calculated.

Since $\cos.^2 A + \sin.^2 A = 1$ (838), we have

$$\cos. 2A = 1 - 2 \sin.^2 A; \quad \text{also, } \cos. 2A = 2 \cos.^2 A - 1.$$

These formulas being true for all angles, $\frac{1}{2}A$ may be substituted for A. Then, transposing,

$$2 \sin.^2 \tfrac{1}{2}A = 1 - \cos. A, \quad \text{and } 2 \cos.^2 \tfrac{1}{2}A = 1 + \cos. A.$$

Therefore,

$$\sin. \tfrac{1}{2}A = \sqrt{\tfrac{1}{2}(1 - \cos. A)},$$

$$\cos. \tfrac{1}{2}A = \sqrt{\tfrac{1}{2}(1 + \cos. A)}, \quad . \quad . \quad . \quad . \quad \text{(IV.)}$$

By these formulas, from the cosine of an angle, may be calculated the sine and cosine of its half, fourth, eighth, etc.

EXERCISES.

848.—1. Demonstrate $\tan. \frac{A}{2} = \frac{\sec. A - 1}{\tan. A}$.

2. What is the value of sin. 15°; cos. 3°; sin. 1° 30′?

FORMULAS FOR LOGARITHMIC USE.

849. In order to render a formula fit for logarithmic calculation, products and quotients must be substituted for sums and differences. This may frequently be done by means of the formulas which follow.

The formulas for the sine and cosine of $(A \pm B)$ become, by adding the third to the first, subtracting the third from the first, adding the second to the fourth, and subtracting the second from the fourth (845),

$$\sin.(A+B)+\sin.(A-B)=2\sin.A\cos.B, \qquad (I.)$$

$$\sin.(A+B)-\sin.(A-B)=2\cos.A\sin.B, \qquad (II.)$$

$$\cos.(A+B)+\cos.(A-B)=2\cos.A\cos.B, \qquad (III.)$$

$$\cos.(A-B)-\cos.(A+B)=2\sin.A\sin.B, \qquad (IV.)$$

In the above, let $A+B=C$, and $A-B=D$; whence, $A=\frac{1}{2}(C+D)$, and $B=\frac{1}{2}(C-D)$. Then,

$$\sin.C+\sin.D=2\sin.\tfrac{1}{2}(C+D)\cos.\tfrac{1}{2}(C-D), \qquad (V.)$$

$$\sin.C-\sin.D=2\cos.\tfrac{1}{2}(C+D)\sin.\tfrac{1}{2}(C-D), \qquad (VI.)$$

$$\cos.C+\cos.D=2\cos.\tfrac{1}{2}(C+D)\cos.\tfrac{1}{2}(C-D), \qquad (VII.)$$

$$\cos.D-\cos.C=2\sin.\tfrac{1}{2}(C+D)\sin.\tfrac{1}{2}(C-D), \qquad (VIII.)$$

By dividing V by VI,

$$\frac{\sin.C+\sin.D}{\sin.C-\sin.D}=\tan.\tfrac{1}{2}(C+D)\cot.\tfrac{1}{2}(C-D)=\frac{\tan.\frac{1}{2}(C+D)}{\tan.\frac{1}{2}(C-D)}.$$

Hence,

$$\sin.C+\sin.D : \sin.C-\sin.D :: \tan.\tfrac{1}{2}(C+D) : \tan.\tfrac{1}{2}(C-D). \qquad (IX.)$$

EXERCISES.

850.—1. Demonstrate $\sin.5A=5\sin.A-20\sin.^3A+16\sin.^5A$.

2. Demonstrate $\sin.(A+B)\sin.(A-B)=\sin.^2A-\sin.^2B$.

TRIGONOMETRICAL TABLES.

851. By the application of algebra to the geometrical principles used in the construction of regular polygons, the student has found that the sine of 30° is $\frac{1}{2}$, and the sine of 18° is $\frac{1}{4}(\sqrt{5}-1)$. From these may be found the cosines of these angles; then (847, IV) the sine and cosine of 15°, and then the sine of 3° (845, III). The sine of 1° may be found as follows:

$$\sin. 3A = \sin. (A + 2A) = \sin. A \cos. 2A + \cos. A \sin. 2A.$$

Substituting the values of cos. 2A and sin. 2A (847),

$$\sin. 3A = 3 \cos.^2 A \sin. A - \sin.^3 A.$$

Hence (838), $\sin. 3A = 3 \sin. A - 4 \sin.^3 A.$

Put 1° for A; then, knowing the value of sin. 3°, and representing the unknown sin. 1° by x,

$$x^3 - \tfrac{3}{4}x + \frac{\sin. 3^\circ}{4} = 0.$$

Only one of the roots of this equation is less than sin. 3°. It must be sin. 1°, and may be calculated by algebraic methods to any required degree of approximation.

Similarly, an equation of the fifth degree, may be formed from the value of sin. 5A; and by its means from the known sin. 1° may be found sin. 12′. Thus, by successive steps, the functions of 1′ and of 1″ may be found to any required degree of accuracy.

Having the sine and cosine of these small angles, the functions of their multiples may be calculated (847). This method, however, is tedious and is not used in practice. It serves to show the possibility of calculating these functions by elementary algebra and geometry. The higher analysis teaches briefer methods.

These numerical functions are called the *natural* sines, tangents, etc., to distinguish them from the logarithmic functions which will be defined presently.

852. The Table of Natural Sines and Tangents gives these functions to six places of figures for every 10′ from 0 to 90°. It also serves as a table of cosines and cotangents.

If the sine or tangent of some intermediate angle is required, it may be found by taking a proportional part of the difference, with as much accuracy as the functions given in the table, except when the angle is nearly a right angle. For example, to find the sine 34° 23′ 30″, the table gives the sine of 34° 20′ = .564007. Since 3′ 30″ is .35 of 10′, multiply 2399, the difference between this sine and that of 34° 30′, by .35, and add the product to the given sine; the sum .564847 is the natural sine of 30° 23′ 30″.

At the beginning of this table, the functions vary with almost perfect uniformity, and in proportion to the angle. Thus, the sine and the tangent of 100′ differ only by one-millionth from one hundred times the sine or the tangent of 1′. At the close of the table, the tangent varies rapidly and the sine varies slowly, and both irregularly. Therefore, for the intermediate angles (those not given in the table), the last lines are less to be relied upon than the first.

The tangent of a large angle may be found with greater accuracy by finding the cotangent of the same angle and taking its reciprocal (837).

LOGARITHMIC FUNCTIONS.

853. Before proceeding to the study of this article, the student should understand the use of the tables of logarithms of numbers.

A *logarithmic* sine, tangent, etc., means the logarithm of the sine, of the tangent, etc. In the tables, the characteristic of every logarithmic trigonometric function is increased by 10. For example, sin. 30° $= \frac{1}{2}$; log. $\frac{1}{2} = \bar{1}.698970$, which is the true logarithm of the sine of 30°; but the tabular logarithmic sine of 30° is 9.698970.

The object of this arrangement is simply to avoid the use of negative characteristics, as would be the case with all the sines and cosines and half of the tangents and cotangents. Therefore, whenever in a calculation, a tabular logarithmic function is added, 10 must be subtracted from the result to find the true logarithm; and whenever a tabular logarithmic function is subtracted, 10 must be added to the result. If, however, in place of subtracting a logarithmic function, the arithmetical complement is added, the result does not need correction, the 10 to be added for one reason, balancing that to be subtracted for the other.

854. The table gives the logarithmic sine, tangent, cosine, and cotangent for every 1′ from 0 to 90°. The degrees are marked at the top of each page and the minutes in the left hand column descending, for the sines and tangents; and the degrees at the bottom of each page and the minutes in the right hand column ascending, for the cosines and cotangents. The columns marked P. P. 1″ contain the proportional part for one second, to facilitate the proper addition or subtraction.

In using the proportional part for the cosine and cotangent, remember that these functions *decrease* when the angle *increases*.

855. *To find the logarithmic sine, etc., of a given angle.*

If the angle is expressed in degrees only, or in degrees and minutes, take the corresponding sine or other function directly from Table IV.

If the angle is expressed in degrees, minutes, and seconds, then take the logarithmic function corresponding to the given degrees and minutes; multiply the proportional part for 1″ by the number of seconds; and add the product to the tabular function, for the sine and tangent, and subtract it for the cosine and cotangent.

For example, to find the tabular logarithmic sine of 40° 13′ 14″

	tab. log. sin. 40° 13′	= 9.810017,
P. P. 1″ = 2.5, . . .	2.5 × 14 . .	= 35,
Therefore, . .	tab. log. sin. 40° 13′ 14″	= 9.810052.

To find the tabular logarithmic cosine of 75° 40′ 21″,

	tab. log. cos. 75° 40′	= 9.393685,
P. P. 1″ = 8.23, . .	8.23 × 21 . .	= 173,
Therefore, . .	tab. log. cos. 75° 40′ 21″	= 9.393512.

This method of using the proportional part given in the tables, gives results that are true to six decimal places, except for the sines, tangents, and cotangents of angles less than three degrees, and for the cosines and cotangents of angles greater than eighty-seven degrees.

The sines and tangents of small angles increase almost uniformly. Therefore, the logarithmic sine and tangent of one of these small angles may be found *nearly*, by adding to the logarithmic sine or tangent of one second the logarithm of the number of seconds in the given angle. This result is subject to the correction in Table V.

The cosines and cotangents of large angles are found in the same way, since they are the sines and tangents of the small angles (841 and 842.)

Since the tangent and cotangent of an angle are reciprocals, the rule just given for finding the tangents of small angles, may be applied to the cotangents also. For the correction, see Table V.

For example, to find the logarithmic sine of $45' \ 23'' = 2723''$,

add to	4.685575,
log. 2723,	3.435048,
	8.120623.
Subtract as in Table V,	13,
tab. log. sin. $45' \ 23''$ =	8.120610.

856. *To find the angle when its logarithmic sine, tangent, cosine, or cotangent is given.*

If the given function is found in Table IV, take the corresponding angle, expressed in degrees, or in degrees and minutes.

If the given function is not in the table, take that which is next less; subtract it from the given function; divide the remainder by the proportional part for $1''$; the quotient is the number of seconds, to be added, in case of sine or tangent, to the angle corresponding to the tabular function used; and to be subtracted in case of the cosine or cotangent.

For example, to find the angle whose tabular logarithmic tangent is 10.456789,

tab. log. tan. $70° \ 44' =$ 10.456501,

P. P. $1'' = 6.75$, ... $288 \div 6.75 = 43$.

Therefore, $70° \ 44' \ 43''$ is the angle sought.

To find the angle whose tabular logarithmic cotangent is 9.876543,

tab. log. cot. $53° \ 3' =$ 9.876326,

P. P. $1'' = 4.38$, ... $217 \div 4.38 = 50$.

Therefore, 53° 2′ 10″ is the angle whose logarithmic cotangent is 9.876543.

When great accuracy is desired and the angle to be found is less than three degrees or greater than eighty-seven, the corrections in Table V may be used, first using Table IV to determine the angle approximately.

RIGHT ANGLED TRIANGLES.

857. The principles have now been established, by which, whenever certain parts of a triangle are known, the remaining parts can be calculated. Since the trigonometrical functions are the ratios between the sides of a right angled triangle, the problems concerning such triangles need no other demonstration than is contained in the definitions.

The sum of the acute angles being 90°, when one is known, the other is found by subtraction.

858. Problem.—*Given the hypotenuse and one angle, to find the other parts.*

The product of the hypotenuse by the sine of either acute angle, is the side opposite that angle. The product of the hypotenuse by the cosine of either acute angle, is the side adjacent to that angle.

859. Problem.—*Given one leg and one angle, to find the other parts.*

The quotient of one leg divided by the sine of the opposite angle is the hypotenuse. The product of one leg by the tangent of the adjacent angle is the other leg.

860. Problem.—*Given one leg and the hypotenuse, to find the other parts.*

The quotient of one leg divided by the hypotenuse is the sine of the angle opposite that leg, and the cosine of the adjacent angle. The other leg may then be found by the previous problem.

861. Problem.—*Given the two legs to find the other parts.*

The quotient of one leg divided by the other is the tangent of the angle opposite the dividend. The hypotenuse may then be found by the second problem.

When, as in the last two problems, two sides are given, the third may be found by the Pythagorean Theorem.

862. Only the sine, cosine, and tangent are used in the above solutions. The student may easily propose solutions by means of the other functions. Since none of the above problems requires addition or subtraction, the operations may all be performed by logarithms.

For example: A railroad track, 463 feet 3 inches long, has a uniform grade of 3°. How high is one end above the other? Here the hypotenuse and one acute angle are given, to find the opposite side.

$$\begin{aligned} \log.\ 463.25 &= 2.665815, \\ \text{tab. log. sin. } 3^\circ &= \underline{8.718800}, \end{aligned}$$

Omitting the tabular 10, the sum 1.384615 is the logarithm of 24.2446. Hence, the ascent is nearly 24 feet 3 inches.

EXERCISES.

863.—1. Construct a figure to illustrate the above, and each of the following.

2. The hypotenuse is 4321, one angle is 25° 30′. Find the other angle and the two legs. Solve this both with and without logarithms.

3. Two posts on the bank of a river are one hundred feet apart; the line joining them is perpendicular to the line from the first post to a certain point on the opposite bank; and the same line makes an angle of $78° 52'$ with the line from the second post to the same point on the opposite bank. How wide is the river?

4. The instrument used in measuring the angle in the above statement is imperfect, the observations being liable to an error of $1'$. To what extent does that affect the calculated result?

5. The hypotenuse being 7093, and one leg 2308.5, find the other leg and the angles.

6. An observer standing 60 feet from a wall measures its angular height, and finds it to be $15° 37'$, his eye being 5 feet from the ground, which is level. How high is the wall?

7. How much would the last result be affected by an error of $5''$ in observing the angle?

8. How much if there had also been an error of 2 inches in measuring the horizontal line?

9. Find the apothegm and radius of a regular polygon of 7 sides, one side being 10 inches.

10. Find the area of a regular dodecagon, the side being 2 feet.

11. The legs being 42.9 and 47.52, find the angles and the hypotenuse.

12. A tower 103 feet high throws a shadow 51.5 feet long upon the level plane; what is the angle of elevation of the sun?

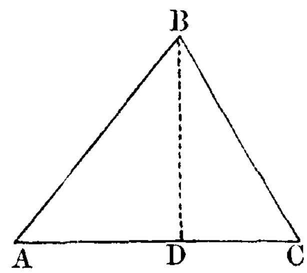
13. How much would the last result be affected by an error of 3 inches in the given height or length?

SOLUTION OF PLANE TRIANGLES.

864. One angle of a triangle being the supplement of the sum of the other two, when two are known the third may be found by subtraction. Also, the sine of either angle is equal to the sine of the sum of the other two.

The letters a, b, and c represent the sides of a triangle respectively opposite the angles A, B, and C.

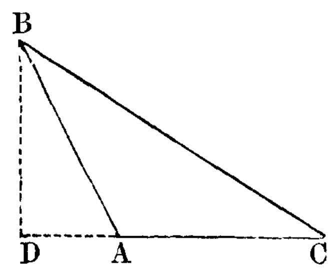
365. Theorem.—*The square of one side of a triangle is equal to the sum of the squares of the other two sides, less twice the product of those sides by the cosine of their included angle.*



For, in the first figure (411),

$$a^2 = b^2 + c^2 - 2b \cdot \text{AD},$$

and in the second figure (412),



$$a^2 = b^2 + c^2 + 2b \cdot \text{AD};$$

but in the first case,

$$\text{AD} = \cos.\ \text{A} \times \text{AB} = c \cos.\ \text{A};$$

and in the second,

$$\text{AD} = -\cos.\ \text{A} \times \text{AB} = -c \cos.\ \text{A}.$$

Substituting these values of AD in their respective equations, both become

$$a^2 = b^2 + c^2 - 2bc \cos.\ \text{A}.$$

By similar reasoning, it may be shown that

$$b^2 = a^2 + c^2 - 2ac \cos.\ \text{B},$$

and $$c^2 = a^2 + b^2 - 2ab \cos.\ \text{C}.$$

These three equations suffice for the solution of all problems on plane triangles, but they are not suitable for logarithmic calculations. The following are not liable to this objection:

866. Theorem.—*Expressing the sum of the sides of any triangle by* p, *then sin.* $\frac{A}{2} = \sqrt{\frac{(\frac{1}{2}p - b)(\frac{1}{2}p - c)}{bc}}$.

For, by the formula just demonstrated,

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Hence (847, IV),

$$\sin. \frac{A}{2} = \sqrt{\tfrac{1}{2}(1 - \cos. A)} = \sqrt{\tfrac{1}{2}\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right)},$$

$$\sin. \frac{A}{2} = \sqrt{\frac{a^2 - b^2 + 2bc - c^2}{4bc}} = \sqrt{\frac{(p - 2b)(p - 2c)}{4bc}},$$

$$\sin. \frac{A}{2} = \sqrt{\frac{(\frac{1}{2}p - b)(\frac{1}{2}p - c)}{bc}}.$$

Similarly, find sin. $\frac{1}{2}$B and sin. $\frac{1}{2}$C in terms of the sides.

The cosine and the tangent may also be expressed in terms of the sides, as follows: By Art. 847, IV,

$$\cos. \frac{A}{2} = \sqrt{\tfrac{1}{2}(1 + \cos. A)} = \sqrt{\tfrac{1}{2}\left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)}.$$

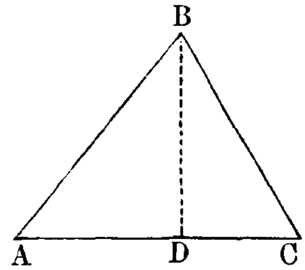
Reducing, $\cos. \frac{A}{2} = \sqrt{\frac{\frac{1}{2}p\,(\frac{1}{2}p - a)}{bc}}$.

Also, $\tan. \frac{A}{2} = \frac{\sin. \frac{1}{2}A}{\cos. \frac{1}{2}A} = \sqrt{\frac{(\frac{1}{2}p - b)(\frac{1}{2}p - c)}{\frac{1}{2}p\,(\frac{1}{2}p - a)}}$.

Similarly, find the cosine and tangent of $\frac{1}{2}$B and of $\frac{1}{2}$C.

867. Theorem.—*The sides of any triangle are proportional to the sines of the opposite angles.*

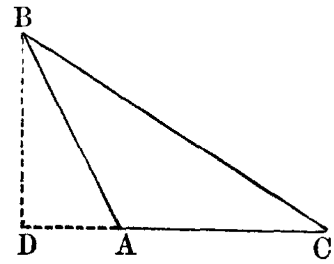
That is, $a : b :: \sin. A : \sin. B$.



For, whether A is acute or obtuse,

$$BD = AB \cdot \sin. A,$$

and $$BD = BC \cdot \sin. C.$$



Therefore, $c \sin. A = a \sin. C$,

and $$a : c :: \sin. A : \sin. C.$$

Similarly, $a : b :: \sin. A : \sin. B$.

868. Theorem.—*One side of a triangle is equal to the sum of the products found by multiplying each of the other sides by the cosine of the angle which it forms with the first side.*

For, $AC = CD \pm DA = BC \cdot \cos. C + BA \cdot \cos. A$.

That is, $b = a \cos. C + c \cos. A$.

869. Theorem.—*The sum of any two sides of a triangle is to their difference as the tangent of half the sum of the two opposite angles is to the tangent of half their difference.*

By Art. 867, $a : b :: \sin. A : \sin. B$.

By composition and division,

$$a + b : a - b :: \sin. A + \sin. B : \sin. A - \sin. B.$$

Hence (849, IX),

$$a + b : a - b :: \tan. \tfrac{1}{2}(A + B) : \tan. \tfrac{1}{2}(A - B.)$$

870. Problem.—*Given the sides of a triangle, to find the angles.*

This rule is derived from the formula for the sine of half an angle (866).

From half the sum of the sides, subtract each of the sides adjacent to the required angle; multiply together these remainders; divide this product by the product of the two adjacent sides, and extract the square root of the quotient. This root is the sine of half the angle sought.

The student may write rules for the solution of this problem from the formulas for cos. $\frac{1}{2}$A, and tan. $\frac{1}{2}$A, and cos. A.

For example, the given sides are $a = 3457$, $b = 4209$, and $c = 6030.4$. For finding all the angles, the formula for the tangent of half an angle is the best, because the same numbers are used for every angle. To find the angle C,

$$\tfrac{1}{2}p = 6848.2 \qquad \tfrac{1}{2}p - b = 2639.2$$
$$\tfrac{1}{2}p - a = 3391.2 \qquad \tfrac{1}{2}p - c = 817.8$$

$$\begin{aligned}
\log.\ (\tfrac{1}{2}p - a)\ . &= 3.530353 \\
\log.\ (\tfrac{1}{2}p - b)\ . &= 3.421472 \\
a.c.\log.\ \tfrac{1}{2}p\ .\ . &= 6.164424 \\
a.c.\log.\ (\tfrac{1}{2}p - c) &= 7.087353 \\
\hline
\text{tab. log. tan.}^2\ \tfrac{1}{2}C &= 20.203602 \\
\text{tab. log. tan.}\ \tfrac{1}{2}C &= 10.101801,
\end{aligned}$$

which is the tab. log. tan. 51° 39′ 16″.4. Therefore, the angle C is 103° 18′ 33″.

In the above calculation, the sum of the logarithms exceeds by 20 the sum required, on account of the arithmetical complement twice used; but the tabular logarithm of tan.^{2} $\frac{1}{2}$C being also 20 more than the true logarithm of tan.^{2} $\frac{1}{2}$C, no correction is necessary.

Find in a similar manner the other two angles, and test the result by comparing the sum with 180°.

There is another method of solving this problem. By dividing any triangle into two right angled triangles, if the sides are known, the altitude and the segments of the base may be found (328). Then the angles may be calculated as in the solutions of right angled triangles.

871. Problem.—*Given two angles and a side, to find the other angle and sides.*

Find the third angle by subtracting the sum of the given two from 180°. Then find the remaining sides by the formula (867),

$$\sin. A : \sin. B :: a : b.$$

872. Problem.—*Given two sides and an angle opposite one of them, to find the other angles and side.*

Find the angle opposite the other given side by the formula,

$$a : b :: \sin. A : \sin. B.$$

Find the third angle by subtraction, and the third side by the formula,

$$\sin. A : \sin. C :: a : c.$$

When the side opposite the given angle is equal to or greater than the other given side, there can be only one solution (287). When it is less than the other given side, there may be two solutions (291 and 300). This is called the ambiguous case. The result is indicated by the trigonometrical formula, for the angle is found by its sine; and for a given sine there are two angles, one acute and one obtuse.

The side opposite the given angle may be so small as to make the triangle impossible (300.) This result is also indicated by the trigonometrical solution, for the sine of the angle sought is found to be greater than unity, which is impossible.

873. Problem.—*Given two sides and the included angle, to find the other angles and side.*

Find the sum of the other angles by subtraction, and the difference of those angles by the formula (869),

$$a + b : a - b :: \tan. \tfrac{1}{2}(A + B) : \tan. \tfrac{1}{2}(A - B).$$

Knowing half the sum and half the difference of the two required angles, take the sum of these two quantities for the greater and their difference for the less of the angles. The third side is found as in the preceding problems.

This problem may be solved, without logarithms, by the formula (865),

$$a^2 = b^2 + c^2 - 2bc \cos. A.$$

AREAS.

874. Theorem.—*The area of a triangle is equal to half the product of any two sides multiplied by the sine of the included angle.*

Thus, the area of triangle ABC $= bc$ sin. A.

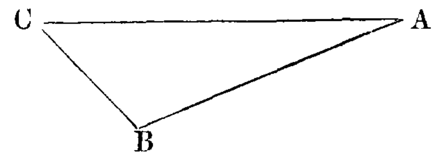
For the altitude BD (see last figure) is the product of the side c by the sine of the angle A.

The student may now review Art. 390.

APPLICATIONS.

875.—1. *To measure the distance from one point to another, when the line between them can not be passed over with the measuring chain or rod.*

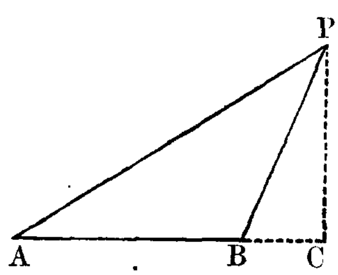
Let A and B be the two points. Take some point C, from which both A and B are visible, and such that the lines AC and BC can be measured with the rod or chain. Measure these and the angle C. Then, in the triangle ABC, two sides and the included angle are known, from which the third side AB can be calculated (873).



If A and B are visible from each other, as when the obstacle between them is open water, then the angles A and B may be observed. In that case it is necessary to measure only one of the sides AC or BC; for, knowing one side and the angles, the other sides may be calculated (871).

2. *To find the height and distance of an inaccessible object.*

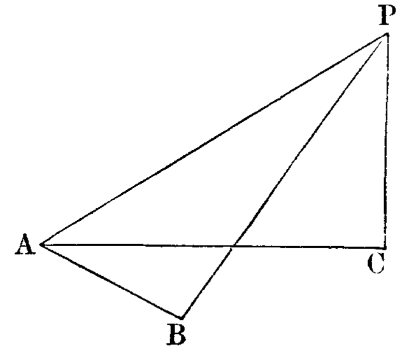
Let P be the top of an object, whose distance from, and height above, the point A are required. At A observe the angle PAC, that is, the angle of inclination of the line AP with the plane of the horizon (537 and 563). Then, measure any length AB, on a horizontal line directly towards the object, and at B observe the angle PBC.



In the triangle APB, the side AB and the angle A are known; also the angle ABP, since it is the supplement of PBC; hence, AP can be calculated. Then $PC = AP \cdot \sin. A$, and $AC = AP \cdot \cos. A$; thus determining the height and distance of the object.

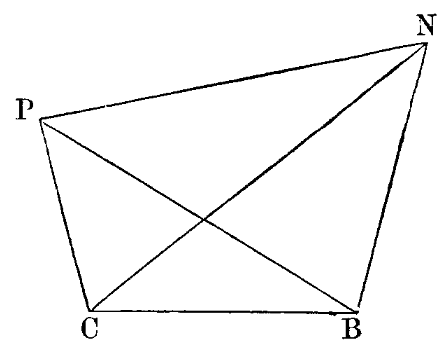
The angle A is called the angular *elevation* of the point P as seen at A, the angle PBC being the elevation of the same point as seen at B. If P were below the level of A, the angle thus observed would be the angular *depression* of the object.

When, as is generally the case, it is inconvenient to measure the line AB "on a horizontal line directly toward the object," measure any length AB in any convenient direction; at A, observe the angle PAB, and the elevation PAC; and at B, observe the angle PBA. Then, in the triangle APB, the side AB and the adjacent angles being known, the side AP may be found, and the height and distance of P calculated as before.



3. *To find the distance between two visible but inaccessible objects.*

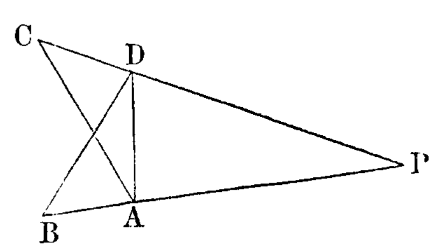
Let P and N be the objects, C and B two accessible points from which both the objects are visible. At C observe the angles PCN and NCB, and if C, B, N, P are not all in the same plane, observe also the angle PCB. At B observe the angles PBC and NBC. Measure CB.



In the triangle PCB, the side CB and its adjacent angles being known, the side CP can be found. In the triangle NCB, the side CB and its adjacent angles being known, the side CN may be found. Then, in the triangle PCN, the sides CP and CN and their included angle being known, the side PN may be found.

4. *To find the width of a river without an instrument for observing angles.*

Let P be a visible point on the further bank, and A a point opposite to it on this side. Take B, C, and D, any convenient accessible points, such that B, A, and P are in a straight line, and C, D, and P are in a straight line; and measure AB, AC, AD, BD, and CD.



All the sides of the triangles ABD and ACD being known, the angles BAD and ADC may be found, and hence their supplements DAP and ADP. Then, from the side AD and the two adjacent angles of the triangle ADP, the side AP may be calculated.

EXERCISES.

376.—1. The sides of a triangle being 70, 80, and 100, what are the angles?

2. Two angles of a triangle are 76° 30′ 23″ and 54° 17′ 51″, and the side opposite the latter is 40.451; find the other sides.

3. Two sides of a triangle are 243.775 and 907.961, and the angle opposite the former is 15° 16′ 17″; find the other parts.

4. Two sides of a triangle are 196.96 and 173.215, and the included angle 40°: find the other angles and side.

5. From a station B, at the base of a mountain, its summit A is seen at an elevation of 60°; after walking one mile towards the summit, up a plane, making an angle of 30° with the horizon to another station C, the angle BCA is observed to be 135°. Find the height of the mountain.

6. Two sides of a parallelogram are 25 and 17.101, and one of its diagonals 38.302; find the other diagonal.

7. A person observing the elevation of a spire to be 35°, advances 80 yards nearer to it, and then finds the elevation is 70°; required the height of the spire.

8. From the top of a tower whose height is 124 feet, the angles of depression of two objects, lying in the same horizontal plane with the base of the tower and in the same direction, are 72° and 48°; what is their distance apart?

CHAPTER XIII.

SPHERICAL TRIGONOMETRY.

877. SPHERICAL TRIGONOMETRY is the investigation of the relations which exist between the sides and angles of spherical triangles.

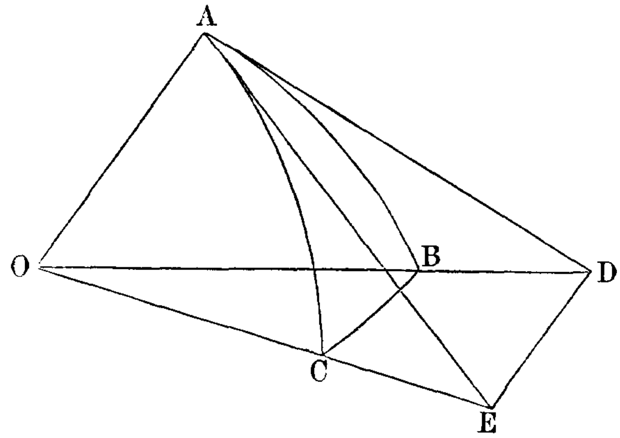
Each side of a spherical triangle being an arc, is the measure of an angle. It has the same ratio to the whole circumference that its angle has to four right angles. It may be measured by degrees, minutes, and seconds, as an angle is measured. It has its sine, tangent, and other trigonometrical functions; it being understood that the sine, etc., of an arc are the sine, etc., of the angle at the center which that arc subtends.

The propositions which express the relations between the sides and angles of a spherical triangle, apply equally well to the faces and diedral angles of a triedral (766 and *seq.*). If the investigation were made from this point of view, as it well might be, the proper title of the subject would be Trigonometry in Space.

THREE SIDES AND AN ANGLE.

878. Theorem.—*The cosine of any side of a spherical triangle is equal to the product of the cosines of the other two sides, increased by the product of the sines of those sides and the cosine of their included angle.*

Let ABC be a spherical triangle, O the center of the sphere, AD and AE tangents respectively to the arcs AB and AC. Thus, the angle EAD is the angle A of the spherical triangle; the angle EOD is measured by the side a, and so on.



From the triangles EOD and EAD (865),

$$\overline{DE}^2 = \overline{OD}^2 + \overline{OE}^2 - 2OD \cdot OE \cos. a,$$

$$\overline{DE}^2 = \overline{AD}^2 + \overline{AE}^2 - 2AD \cdot AE \cos. A.$$

By subtraction, the triangles OAE and OAD being right angled,

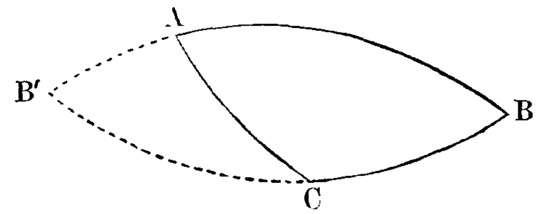
$$0 = 2\overline{OA}^2 + 2AD \cdot AE \cos. A - 2OD \cdot OE \cos. a;$$

Therefore, $$\cos. a = \frac{OA}{OE} \cdot \frac{OA}{OD} + \frac{AE}{OE} \cdot \frac{AD}{OD} \cos. A;$$

that is, $$\cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. A.$$

In the above construction, the sides which contain the angle A are supposed less than quadrants, since the tangents at A meet OB and OC produced. That the formula just demonstrated is true when these sides are not less than quadrants, is shown thus:

Suppose *one* of the sides greater than a quadrant, for example, AB. Produce BA and BC to B′, and represent AB′ and CB′ by c' and a' respectively.



Then, in the triangle AB′C, as just demonstrated,

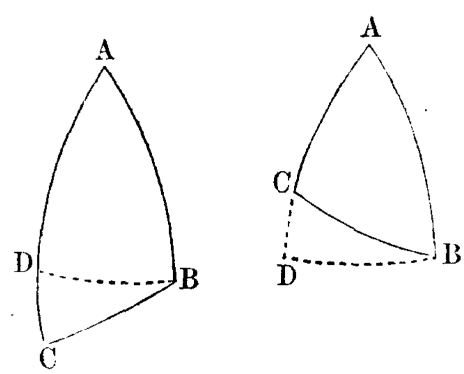
$$\cos. a' = \cos. b \cos. c' + \sin. b \sin. c' \cos. \text{B}'\text{AC}.$$

Now, a', c', and B′AC are respectively supplements of a, c, and BAC. Hence,

$$\cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. \text{A}.$$

When *both* the sides which contain the angle A are greater than quadrants, produce them to form the auxiliary triangle, and the demonstration is similar to the last.

Suppose that one of the sides b and c is a quadrant, for example, c. On AC, produced if necessary, take AD equal to a quadrant, and join BD. Now A is a pole of the arc BD (754), and therefore that arc measures the angle A (760).



Then, from the triangle BCD,

$$\cos. a = \cos. \text{CD} \cos. \text{BD} + \sin. \text{CD} \sin. \text{BD} \cos. \text{CDB};$$

but —CD is the complement of b, BD measures A, and CDB is a right angle. Hence, this equation becomes,

$$\cos. a = \sin. b \cos. A,$$

and the formula to be demonstrated reduces to this, when c is a quadrant.

The proposition having been demonstrated for any angle of any spherical triangle,

$$\cos. b = \cos. a \cos. c + \sin. a \sin. c \cos. B,$$

$$\cos. c = \cos. a \cos. b + \sin. a \sin. b \cos. C.$$

These have been called the fundamental equations of Spherical Trigonometry. By their aid, when any three of the elements of a spherical triangle are known, the others may be calculated.

A SIDE AND THE THREE ANGLES.

879. Since the formulas just demonstrated are true of all spherical triangles, they apply to the polar triangle of any given triangle. Therefore, denoting the sides and angles of the polar triangle, by accenting the letters of their corresponding parts in the given triangle,

$$\cos. a' = \cos. b' \cos. c' + \sin. b' \sin. c' \cos. A',$$

but $a' = 180° - A$, $b' = 180° - B$, and $A' = 180° - a$, etc. (777). Substituting these values of a', b', etc.,

$$\cos. (180° - A) = \cos. (180° - B) \cos. (180° - C) + \sin. (180° - B) \sin. (180° - C) \cos. (180° - a).$$

Reducing (829), and changing the signs,

$$\cos. A = -\cos. B \cos. C + \sin. B \sin. C \cos. a.$$

Similarly,

$$\cos. B = -\cos. A \cos. C + \sin. A \sin. C \cos. b,$$

$$\cos. C = -\cos. A \cos. B + \sin. A \sin. B \cos. c.$$

None of the above formulas is suited for logarithmic calculation.

FORMULAS FOR LOGARITHMIC USE.

880. Let p represent the perimeter, that is, $p = a + b + c$.

By transposing and dividing the fundamental formula (878),

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}.$$ Therefore (845, IV),

$$1 - \cos. A = \frac{\sin. b \sin. c + \cos. b \cos. c - \cos. a}{\sin. b \sin. c.} = \frac{\cos. (b-c) - \cos. a}{\sin. b \sin. c.}.$$

Substituting for this numerator its value (849, VIII), and dividing by 2,

$$\tfrac{1}{2}(1 - \cos. A) = \frac{\sin. \frac{1}{2}(a + b - c) \sin. \frac{1}{2}(a - b + c)}{\sin. b \sin. c.}.$$

Substituting p for its value, and extracting the root (847, IV),

$$\sin. \frac{A}{2} = \sqrt{\frac{\sin. (\frac{1}{2}p - b) \sin. (\frac{1}{2}p - c)}{\sin. b \sin. c.}}.$$

To find the value of the cosine of half the angle,

$$1+\cos.A=\frac{\sin.b\ \sin.c-\cos.b\ \cos.c+\cos.a}{\sin.b\ \sin.c}=\frac{\cos.a-\cos.(b+c)}{\sin.b\ \sin.c}.$$

Hence, $$\cos.\frac{A}{2}=\sqrt{\frac{\sin.\frac{1}{2}p\ \sin.(\frac{1}{2}p-a)}{\sin.b\ \sin.c}}.$$

Dividing sin. $\frac{1}{2}$A by cos. $\frac{1}{2}$A,

$$\tan.\frac{A}{2}=\sqrt{\frac{\sin.(\frac{1}{2}p-b)\ \sin.(\frac{1}{2}p-c)}{\sin.\frac{1}{2}p\ \sin.(\frac{1}{2}p-a)}}.$$

Find the analogous formulas for the sine, cosine, and tangent of $\frac{1}{2}$B and of $\frac{1}{2}$C.

881. Let E represent the spherical excess, that is, $E=A+B+C-180°$.

By reasoning upon the polar triangle as in the preceding article, the formula for the sine of half an angle becomes

$$\sin.\frac{180°-a}{2}=\sqrt{\frac{\sin.\frac{1}{2}(180°-A+B-C)\ \sin.\frac{1}{2}(180°-A-B+C)}{\sin.(180°-B)\ \sin.(180°-C)}};$$

but $\quad \sin.\frac{1}{2}(180°-a)=\sin.(90°-\frac{1}{2}a)=\cos.\frac{1}{2}a,$

and $\quad \sin.\frac{1}{2}(180°-A+B-C)=\sin.(B-\frac{1}{2}E)$, etc.

Therefore, $$\cos.\frac{a}{2}=\sqrt{\frac{\sin.(B-\frac{1}{2}E)\ \sin.(C-\frac{1}{2}E)}{\sin.B\ \sin.C}}.$$

Similarly, from the formula for the cosine of half the angle,

$$\sin.\frac{a}{2}=\sqrt{\frac{\sin.\frac{1}{2}E\ \sin.(A-\frac{1}{2}E)}{\sin.B\ \sin.C}}.$$

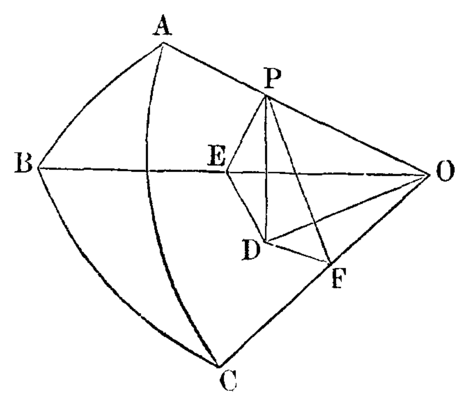
Hence, $$\tan.\frac{a}{2}=\sqrt{\frac{\sin.\frac{1}{2}E\sin.(A-\frac{1}{2}E)}{\sin.(B-\frac{1}{2}E)\sin.(C-\frac{1}{2}E)}}.$$

Since E must be less than 360° (771), sin. $\frac{1}{2}$E is positive; and since sin. $\frac{1}{2}a$ is a real quantity, sin. (A — $\frac{1}{2}$E) must be positive. Therefore, any angle of a spherical triangle is greater than half the spherical excess.

OPPOSITE SIDES AND ANGLES.

882. Theorem.—*The sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.*

Let ABC be the spherical triangle, and O the center of the sphere. From any point P in OA, let PD fall perpendicular to the plane BOC; make DE, DF perpendicular respectively to BO, OC; and join PE, PF, and OD.



The plane PED is perpendicular to the plane BOC (556). Therefore, OE is perpendicular to the plane PED, the angle PED is the same as the angle B (759), and PEO is a right angle. Therefore, PE = OP · sin. POE = OP · sin. c; and PD = PE · sin. B = OP · sin. c sin. B.

Similarly, PD = OP · sin. b sin. C;

therefore, OP · sin. c sin. B = OP · sin. b sin. C.

$$\frac{\sin.B}{\sin.C}=\frac{\sin.b}{\sin.c}, \text{ or } \sin.B : \sin.C :: \sin.b : \sin.c.$$

The figure supposes b, c, B, and C to be each less than 90°. When this is not the case, the figure and the demonstration are slightly modified. For example, when B is greater than a right angle, the point D falls beyond BO, and PED becomes the supplement of B, having the same sine.

FOUR CONTIGUOUS PARTS.

883. Theorem.—*The product of the cotangent of one side by the sine of another, is equal to the product of the cosine of the included angle by the cosine of the second side, plus the product of the sine of the included angle by the cotangent of the angle opposite the first side.*

We have (878 and 882),

$$\cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. A,$$

$$\cos. c = \cos. a \cos. b + \sin. a \sin. b \cos. C,$$

$$\sin. c = \frac{\sin. a \sin. C}{\sin. A}.$$

Eliminate c by substituting these values of $\cos. c$ and $\sin. c$ in the first equation,

$$\cos. a = (\cos. a \cos. b + \sin. a \sin. b \cos. C) \cos. b + \frac{\sin. a \sin. b \cos. A \sin. C}{\sin. A};$$

transposing and reducing, since $1 - \cos.^2 b = \sin.^2 b$,

$$\cos. a \sin.^2 b = \sin. a \sin. b \cos. b \cos. C + \sin. a \sin. b \cot. A \sin. C;$$

dividing by $\sin. a \sin. b$,

$$\cot. a \sin. b = \cos. b \cos. C + \cot. A \sin. C.$$

The demonstration being general, may be applied to other angles and sides, making these five additional formulas:

$$\cot. b \sin. a = \cos. a \cos. C + \cot. B \sin. C,$$
$$\cot. b \sin. c = \cos. c \cos. A + \cot. B \sin. A,$$
$$\cot. c \sin. b = \cos. b \cos. A + \cot. C \sin. A,$$
$$\cot. c \sin. a = \cos. a \cos. B + \cot. C \sin. B,$$
$$\cot. a \sin. c = \cos. c \cos. B + \cot. A \sin. B.$$

FORMULAS OF DELAMBRE.

884. Putting $\frac{1}{2}$A and $\frac{1}{2}$B for A and B respectively, in formula I, Art. 845,

$$\sin. \tfrac{1}{2}(A + B) = \sin. \tfrac{1}{2}A \cos. \tfrac{1}{2}B + \cos. \tfrac{1}{2}A \sin. \tfrac{1}{2}B.$$

Substitute the values of the factors of the second member, as found in Art. 880,

$$\sin. \frac{A+B}{2} = \frac{\sin.(\frac{1}{2}p-a)+\sin.(\frac{1}{2}p-b)}{\sin. c}\sqrt{\frac{\sin.\frac{1}{2}p \sin. (\frac{1}{2}p-c)}{\sin. a \sin. b}};$$

but,

$$\sin.(\tfrac{1}{2}p-a)+\sin.(\tfrac{1}{2}p-b) = \sin.\left(\frac{c}{2}-\frac{a-b}{2}\right)+\sin.\left(\frac{c}{2}+\frac{a-b}{2}\right),$$

(849, I), ... $= 2 \sin. \frac{1}{2}c \cos. \frac{1}{2}(a-b)$,

and (847, I), $\sin. c = 2 \sin. \frac{1}{2}c \cos. \frac{1}{2}c.$

Substituting these values, also cos. $\frac{1}{2}$C for the radical (880),

$$\sin. \frac{A + B}{2} = \frac{\cos. \frac{1}{2}(a - b)}{\cos. \frac{1}{2}c} \cos. \tfrac{1}{2}C,$$

or, $$\frac{\sin. \frac{1}{2}(A + B)}{\cos. \frac{1}{2}C} = \frac{\cos. \frac{1}{2}(a - b)}{\cos. \frac{1}{2}c}.$$

Similarly, by beginning with formulas II, III, and IV of Art. 845, we find,

$$\frac{\sin.\ \frac{1}{2}(A - B)}{\cos.\ \frac{1}{2}C} = \frac{\sin.\ \frac{1}{2}(a - b)}{\sin.\ \frac{1}{2}c},$$

$$\frac{\cos.\ \frac{1}{2}(A + B)}{\sin.\ \frac{1}{2}C} = \frac{\cos.\ \frac{1}{2}(a + b)}{\cos.\ \frac{1}{2}c},$$

$$\frac{\cos.\ \frac{1}{2}(A - B)}{\sin.\ \frac{1}{2}C} = \frac{\sin.\ \frac{1}{2}(a + b)}{\sin.\ \frac{1}{2}c}.$$

These four formulas of Delambre were published by him in 1807.

NAPIER'S ANALOGIES.

885. Divide the first of the formulas of Delambre by the third, the second by the fourth, then the fourth by the third, and the second by the first, and these results are obtained:

$$\frac{\tan.\ \frac{1}{2}(A + B)}{\cot.\ \frac{1}{2}C} = \frac{\cos.\ \frac{1}{2}(a - b)}{\cos.\ \frac{1}{2}(a + b)},$$

$$\frac{\tan.\ \frac{1}{2}(A - B)}{\cot.\ \frac{1}{2}C} = \frac{\sin.\ \frac{1}{2}(a - b)}{\sin.\ \frac{1}{2}(a + b)},$$

$$\frac{\tan.\ \frac{1}{2}(a + b)}{\tan.\ \frac{1}{2}c} = \frac{\cos.\ \frac{1}{2}(A - B)}{\cos.\ \frac{1}{2}(A + B)},$$

$$\frac{\tan.\ \frac{1}{2}(a - b)}{\tan.\ \frac{1}{2}c} = \frac{\sin.\ \frac{1}{2}(A - B)}{\sin.\ \frac{1}{2}(A + B)}.$$

These formulas may be stated as proportions, and are called Napier's Analogies, from their inventor, analogy being formerly used as synonymous with proportion.

886. In the first of the above equations, cos. $\frac{1}{2}(a-b)$ and cot. $\frac{1}{2}$C are necessarily positive; hence, tan. $\frac{1}{2}(A+B)$ and cos. $\frac{1}{2}(a+b)$ are of the same sign; thus, $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ are either both less or both greater than ninety degrees.

In the second of the above equations, sin. $\frac{1}{2}(a+b)$ and cot. $\frac{1}{2}$C are positive; hence, tan. $\frac{1}{2}(A-B)$ and sin. $\frac{1}{2}(a-b)$ have the same sign; thus, $\frac{1}{2}(A-B)$ and $\frac{1}{2}(a-b)$ are either both positive, both negative, or both zero. Therefore, in any spherical triangle, the greater angle is opposite the greater side, and conversely.

EXERCISES.

887.—1. Find the formula that results from applying the principle of polar triangles to the first of Napier's Analogies; also, to the first formula of Art. 883.

2. State a theorem applying the principle of Art. 878 to triedrals.

3. Show, from the third of Napier's Analogies, that the sum of any two sides of a spherical triangle is greater than the third.

RIGHT ANGLED SPHERICAL TRIANGLES.

888. The foregoing formulas may be applied to right angled triangles by supposing one of the angles to be right, for example A. In this manner we have:

Art. 878, 1st formula,	$\cos. a = \cos. b \cos. c,$	(I.)
Art. 879, 1st formula,	$\cos. a = \cot. B \cot. C,$	(II.)
Art. 882,	$\sin. b = \sin. a \sin. B$	(III.)
" "	$\sin. c = \sin. a \sin. C$	
Art. 883, 1st formula,	$\tan. b = \tan. a \cos. C$	(IV.)
" " 6th formula,	$\tan. c = \tan. a \cos. B$	

Art. 883, 3rd formula, $\left.\begin{array}{l}\tan. b = \sin. c \tan. B \\ \tan. c = \sin. b \tan. C\end{array}\right\},$ (v.)
" " 4th formula,

Art. 879, 2nd formula, $\left.\begin{array}{l}\cos. B = \sin. C \cos. b \\ \cos. C = \sin. B \cos. c\end{array}\right\}.$ (vi.)
" " 3rd formula,

In deducing ii, iv, and v, the formulas are reduced somewhat by divisions. These are sufficient for the solution of every case. These principles may be stated as follows:

cos. hyp. = product of cosines of sides,

cos. hyp. = product of cotangents of angles,

sine side = sine opposite angle × sine hyp.,

tan. side = tan. hyp. × cosine included angle,

tan. side = tan. opposite angle × sine other side,

cos. angle = cos. opposite side × sine other angle.

889. Since the cosine of the hypotenuse has the same sign as the product of the cosines of the other two sides, it follows either that two of these three cosines are negative, or none. Therefore, in a right angled spherical triangle, either all the sides are less than quadrants, or two are greater and one is less.

It appears also (v) that the tangent of an oblique angle and of its opposite side have the same sign. Therefore, these two parts of the triangle are either both less or both greater than 90°. This is expressed by saying they are of the same *species*.

NAPIER'S RULE OF CIRCULAR PARTS.

890. A mnemonic rule for the formulas of right angled spherical triangles was invented by Napier, and published with his description of logarithms in 1614.

The right angle being omitted, five parts of the triangle remain. The two sides which include the right angle, the complements of the other angles, and the complement of the hypotenuse are called the *circular parts* of the triangle. These are supposed to be arranged around a circle in the order they occur in the triangle. Any one of the five circular parts may be called the *middle* part, then the two next to it are the *adjacent* parts, and the remaining two are the *opposite* parts.

Napier's rule is: The sine of the middle part is equal to the product of the tangents of the adjacent parts, also to the product of the cosines of the opposite parts.

The words sine and middle having their first vowel the same, also the words tangent and adjacent, also the words cosine and opposite, renders this rule very easy to remember. For example, if the complement of the hypotenuse be the middle part, then the complements of the angles are the adjacent parts, and the sides are the opposite parts; this gives formulas I and II.

SOLUTION OF RIGHT ANGLED TRIANGLES.

891. Problem.—*Given the hypotenuse and an oblique angle, to find the other angle and the sides.*

Find the other oblique angle by formula II, the side opposite the given angle by III, and the adjacent side by IV.

For example, given the hypotenuse 64° 17′ 35″, and an angle 70°, to find the opposite side,

$$\begin{aligned}\text{tab. log. sin. } 70^\circ \quad . \quad . &= 9.972986,\\ \text{tab. log. sin. } 64^\circ\, 17'\, 35'' &= \underline{9.954737,}\\ \text{tab. log. sin. } 57^\circ\, 51'\, 11'' &= 9.927723.\end{aligned}$$

Therefore, the required side is 57° 51′ 11″. It is known to be acute because its opposite angle is acute (889).

892. Problem.—*Given one side and the adjacent oblique angle, to find the other sides and angle.*

Find the hypotenuse by IV, the other side by V, and the other angle by VI.

893. Problem.—*Given the two sides, to find the hypotenuse and angles.*

Find the hypotenuse by I, and the angles by V.

894. Problem.—*Given the hypotenuse and one side, to find the angles and the other side.*

Find the included angle by IV, the other side by I, and the remaining angle by III.

895. Problem.—*Given the two oblique angles, to find the three sides.*

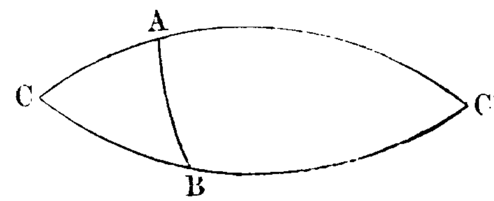
Find the hypotenuse by II, and the other sides by VI.

In the above solutions there is no ambiguous case. Whenever a part is found by means of its sine, its species is determined by the principle of Art. 889. In the 1st and 4th problems, if the given parts are both of 90°, the triangle is indeterminate. The student may show why.

896. Problem.—*Given a side and its opposite angle, to find the other sides and angle.*

Find the hypotenuse by III, the other side by V, and the other angle by VI.

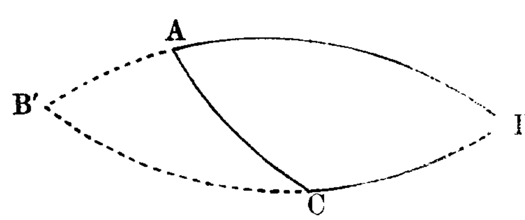
Here the triangle is ambiguous, as all the parts are found by their sines. Suppose BAC to be a triangle right angled at A, and that C and c are the given parts. Produce CB and CA to meet in C′. Then the triangle C′AB has the same conditions as the given triangle, for it has a right angle at A, the given side BA, and C′ = C, the given angle.



897. The solution of an oblique triangle may be made in some cases to depend immediately upon the solution of a right angled triangle. If a triangle has one of its sides a quadrant, then its polar triangle has its corresponding angle a right angle. The polar triangle can be solved by the preceding methods, and thus the elements of the primitive triangle become known.

If a triangle is isosceles, an arc from the vertex to the middle point of the base divides it into two equal right angled triangles, by the solution of which the elements of the isosceles triangle are found.

If a triangle has two sides supplementary, as b and c, the sides a and c may be produced to B′, making the isosceles triangle B′AC, which may be solved as above, giving the elements of the original triangle.



If a triangle has two of its angles supplementary, then its polar triangle has two of its sides supplementary. This may be studied in the manner just stated, and thus the parts of the primitive triangle become known.

EXERCISES.

898.—1. Show that in a right angled spherical triangle, a side is less than its opposite angle when both are acute, and greater when both are obtuse.

2. The sides are 57° 51′ 8″ and 35° 23′ 30″; find the hypotenuse and the angles.

3. The hypotenuse is 71° 39′ 37″ and one angle 79° 56′ 4″; find the sides and the other angle.

4. One side is 140°, the opposite angle is 138° 14′ 14″; find the remaining parts.

5. Show that if the hypotenuse is 90°, one of the sides must be 90°, and conversely.

6. The sides are 90°, 76° 49′ 55″, 41° 45′ 46″; find the angles.

7. A lateral edge of a pyramid whose base is a square, makes angles of 60° and 65° respectively with the two conterminous sides of the base; find the diedral angle of that edge.

SOLUTION OF SPHERICAL TRIANGLES.

899. Problem.—*Given the sides, to find the angles.*

Either of the angles may be found by the formulas of Art. 880. When all the angles are required, the formula for the tangent is to be preferred.

900. Problem.—*Given the angles, to find the sides.*

Either of the sides may be found by the formulas of Art. 881.

901. Problem.—*Given two sides and the included angle, to find the other angles and side.*

The half sum of the other angles may be found by the first of Napier's Analogies, and the half difference by the second; and hence, the angles themselves. Then the third side may be found by the proportion of Art. 882. If the ambiguity attendant upon the use of the sine is not removed by observing that the greater side of a triangle is always opposite the greater angle (886), then the third side may be found by Art. 881, or by the third or fourth of Napier's Analogies, or by one of the formulas of Delambre.

For example, given the side $a = 76° 35' 36''$, $b = 50° 10' 30''$, and the angle $C = 34° 15' 3''$.

By the 1st analogy,

$$\tan. \tfrac{1}{2}(A + B) = \cot. \tfrac{1}{2}C \frac{\cos. \frac{1}{2}(a - b)}{\cos. \frac{1}{2}(a + b)}.$$

$$\begin{array}{lcr}
\text{tab. log. cot. } \tfrac{1}{2}C \quad . \quad . \quad . & = & 10.511272 \\
\text{tab. log. cos. } \tfrac{1}{2}(a - b) \quad . & = & 9.988355 \\
a.\ c.\ \text{tab. log. cos. } \tfrac{1}{2}(a + b) & = & \underline{0.348717} \\
\text{tab. log. tan. } \tfrac{1}{2}(A + B) & = & 10.848344 \\
\therefore \tfrac{1}{2}(A + B) & = & 81° 55' 47''
\end{array}$$

By the 2nd analogy,

$$\tan. \tfrac{1}{2}(A - B) = \cot. \tfrac{1}{2}C \frac{\sin. \frac{1}{2}(a - b)}{\sin. \frac{1}{2}(a + b)}.$$

$$\begin{array}{lcr}
\text{tab. log. cot. } \tfrac{1}{2}C \quad . \quad . \quad . & = & 10.511272 \\
\text{tab. log. sin. } \tfrac{1}{2}(a - b) \quad . & = & 9.358899 \\
a.\ c.\ \text{tab. log. sin. } \tfrac{1}{2}(a + b) & = & \underline{0.048648} \\
\text{tab. log. tan. } \tfrac{1}{2}(A - B) & = & 9.918819 \\
\therefore \tfrac{1}{2}(A - B) & = & 39° 40' 33''
\end{array}$$

Hence, $A = 121° 36' 20''$,

and $B = 42° 15' 14''$.

Since the remaining side must be less than either of the given sides, it may be found by the proportion,

$$\sin. A : \sin. C :: \sin. a : \sin. c;$$

or by the 4th analogy, as follows:

$$\tan. \tfrac{1}{2}c = \tan. \tfrac{1}{2}(a - b)\frac{\sin. \frac{1}{2}(A + B)}{\sin. \frac{1}{2}(A - B)}$$

$$\begin{array}{ll} \text{tab. log. tan. } \tfrac{1}{2}(a - b) \quad . & = 9.370544 \\ \text{tab. log. sin. } \tfrac{1}{2}(A + B) \quad . & = 9.995677 \\ a.\ c. \text{ tab. log. sin. } \tfrac{1}{2}(A - B) & = \underline{.194877} \\ \text{tab. log. tan. } \tfrac{1}{2}c \quad . \quad . \quad . & = 9.561098 \end{array}$$

$$\therefore \tfrac{1}{2}c = 20° \ 0' \ 5'', \text{ and } c = 40° \ 0' \ 10''.$$

902. Problem.—*Given one side and the adjacent angles, to find the other sides and angle.*

The half sum of the other sides may be found by the 3rd analogy, and the half difference by the 4th; and hence, the sides themselves. Then the third angle may be found by the proportion of Art. 882.

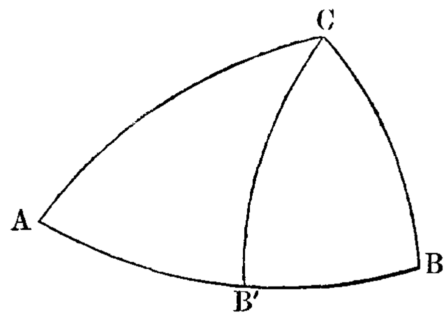
If the ambiguity attendant upon the use of the sine is not removed by observing that the greater angle is opposite the greater side, then it may be found by Art. 880, or by the 1st or 2nd analogy, or by one of the formulas of Delambre.

903. Problem.—*Given two sides and an angle opposite one of them, to find the other angles and side.*

The angle opposite the other given side may be found by Art. 882, and then the remaining angle and side from Napier's Analogies.

Since the sine is used to find the first angle, there may be two solutions. The ambiguity is sometimes removed by observing that the greater angle is opposite the greater side. When only one value of the angle found from its sine is consistent with this principle, there is but one solution.

When both values of the angle thus found are consistent with this principle, there are two solutions, that is, there are two distinct spherical triangles which have the given elements. When the angle A and the sides a and b are given, b being greater than a, if both values found for B are greater than A, then there are two triangles, ABC and AB′C, which have the given sides and angle.



When the same parts are given, and b is less than a, if both values found for B are less than A, there are two solutions. In this case the given angle must have been obtuse, and in the former case it must have been acute.

It may happen that neither value of the angle found from its sine is consistent with the principle stated. This shows that the given conditions are incompatible, and that the triangle is impossible.

904. Problem.—*Given two angles and a side opposite one of them, to find the other sides and angle.*

The side opposite the other given angle may be found by the proportion of Art. 882, and then the remaining angle and side from Napier's Analogies, as in the preceding solution.

This case is precisely analogous to the last; it presents the same ambiguity, and the ambiguity is resolved in the same manner.

EXERCISES.

905.—1. The sides are 60° 4′ 54″, 135° 49′ 20″, and 146° 37′ 15″; find the angles.

2. Find the diedral angle of a regular tetraedron.

3. The sides are 105°, 90°, and 75°; find the sines of the angles without the use of the tables.

4. The angles are 32° 26′ 7″, 36° 45′ 28″, and 130° 5′ 23″; find the three sides.

5. Two sides are 70° and 80°, and the included angle 130°; find the remaining angles and side.

6. Two sides are 89° 16′ 54″ and 52° 39′ 5″, the angle opposite the former is 70° 39′; find the remaining parts.

7. Given the latitude of Paris 48° 50′ 12″, the latitude of New York 40° 17′ 17″, and the longitude of New York west of Paris 76° 20′ 27″, to find the distance between these points, along an arc of a great circle; the earth being considered a sphere of a radius of 3956 miles.

8. How much would the last result be affected by an error of 2″ in the given longitude? in one of the given latitudes?

CHAPTER XIV.

LOGARITHMS.

906. Nearly all trigonometrical calculations are made by means of logarithms. To understand this chapter, the student must be acquainted with the algebraic theory of positive and negative exponents. He may refer to the algebra for an investigation of the principles and the methods of calculating tables.

COMMON LOGARITHMS.

907. The COMMON LOGARITHM of a number is the exponent of that power of 10 which is equal to the number. Hence,

The logarithm	"	of	10	is	1,
"	"	"	100	"	2,
"	"	"	1000	"	3, etc.

Again,

the logarithm		of	1	is	0,
"	"	"	$\frac{1}{10}$ or .1	"	—1,
"	"	"	$\frac{1}{100}$ or .01	"	—2, etc.

Numbers greater than unity have positive logarithms; numbers less than unity have negative logarithms. The powers of 10 have the positive integers for their logarithms, and the reciprocals of those powers have the negative integers for their logarithms. No other numbers have integral logarithms. That part of a logarithm which is not integral is always expressed by decimals.

CHARACTERISTIC.

908. The CHARACTERISTIC of a logarithm is its integral part.

The MANTISSA of a logarithm is the decimal part.

For convenience of calculation, it is an established rule that the mantissa of a logarithm is always positive, and only the characteristic of a negative logarithm is negative. To express this, the negative sign is written over the characteristic. Thus,

$$\log.\ .2 = \bar{1}.301030 = -1 + .301030,$$
$$\log\ .08 = \bar{2}.903090 = -2 + .903090.$$

If any number is between 1 and 10, its logarithm is between 0 and 1; if a number is between 10 and 100, its logarithm is between 1 and 2, and so on; the characteristic of the logarithm is always one less than the number of integral places in the given number. If the number is between 1 and .1, its logarithm is between 0 and -1; hence, its characteristic is -1. If the number is between .1 and .01, its logarithm is between -1 and -2; hence, its characteristic is -2, and so on. The characteristic of the logarithm of a fraction is numerically one more than the number of ciphers between the decimal point and the first significant figure of the given fraction written decimally.

The student who has learned the theory of algebraic signs will perceive that the above rules are included in the following:

The characteristic of the logarithm denotes how many places the first significant figure of the number is to the left of the unit's place.

The characteristics of logarithms are not given in the tables, but must be found as above. If this rule be taken conversely, it shows how to place the decimal point, when the number is found from its given logarithm.

TABLE OF LOGARITHMS.

909. Let c represent the characteristic and d the mantissa of any logarithm, and let N represent the number.

By the definition, $10^{c+d} = \text{N}$.

Multiplying by 10, $10^{c+1+d} = 10\text{N}$.

That is, if $c + d$ is the logarithm of N, $c + 1 + d$ is the logarithm of 10N, the mantissa of each being d. Hence, multiplying a number by 10 does not change the mantissa of its logarithm, and it is the same when the number is multiplied or divided by any power of 10. In other words: if two numbers have the same significant figures, their logarithms have the same mantissas.

For example,

$$\log.\ 5 = .698970,$$

$$\log.\ 5000 = 3.698970,$$

$$\log.\ .005 = \bar{3}.698970.$$

The table in this work gives the mantissa of the logarithm of every number from 1000 to 11000. It follows that the mantissa of the logarithm of every number less than 11000 may be found in the table.

The first three or four figures of each number are given in the left hand column (see Table); the next figure, at the head and at the foot of the several columns of mantissas. The mantissas in the column under 0 are given to six decimal places. The first and second decimal figures of this column are understood to be repeated across the page, and for the spaces in the lines below. In the remaining columns, 1 to 9, only the last four of the six decimal figures of each mantissa are given.

When the second decimal figure changes from 9 to 0, the remaining mantissas of the line are marked, to indicate that, in these cases, the first two decimal figures are taken from the line below.

The last column contains the difference between two successive mantissas, called the *tabular difference.*

In all cases, the mantissa is only an approximation. The large tables of *Adrien Vlacq* give the logarithms to ten places of decimals of all numbers from 1 to 100000. The last figure is given within one-half a unit of its own order; that is, if the first figure of the part not given is 5 or more, then the last figure given is increased by 1.

TO FIND THE LOGARITHM OF A GIVEN NUMBER.

910. If the significant figures of the number are the same as those of any number between 1000 and 11000, find the mantissa in the table and prefix the proper characteristic.

For example, to find the logarithm of 1245, find 124 in column N; in the same line and in column 5, find 5169; prefix .09 from column 0; then prefix the characteristic 3; and the logarithm of 1245 is 3.095169. Similarly,

$$\log.\ \ 124500 = 5.095169,$$
$$\log.\ .0001245 = \bar{4}.095169.$$

If the significant figures are those of a number less than 1000, annex ciphers to make a number between 1000 and 11000, and proceed as before. For example, the logarithm of 16 has the same mantissa as the logarithm of 1600, which is .204120. Therefore, the logarithm of 16 is. 1.204120.

If the significant figures of the given number occupy more places than the numbers in the table, find the mantissa for the first four or five figures; regard the remaining figures as a decimal fraction, and add to the mantissa already found the proportional part of the tabular difference.

For example, to find the logarithm of 3.1416.

The mantissa of log. 3141 is 497068,
six-tenths of the tabular difference, 138, is 83,
the characteristic being 0, ...497151 is the logarithm sought. It is assumed that the mantissa of the logarithm of 3141.6 is the same as of 3141 increased by six-tenths of the difference between the mantissas of 3141 and 3142.

To find the logarithm of 365.242.

The mantissa of log. 3652 is = 562531,
tab. diff. = 119; 119 × .42 = 50.
Therefore, log. 365.242 = 2.562581.

All figures beyond the six places of decimals are rejected from the calculations, taking care that the last figure used shall be the nearest. Thus, six-tenths of 138 is nearer to 83 than to 82.

When the tabular difference varies rapidly, as at the beginning of the table, there may be slight errors in its use, for the logarithms do not vary as the numbers. On this account, for all numbers between 10000 and 11000, it is better to use the last two pages of the Table instead of the first ten lines.

If the given number has more than six significant figures, the seventh and subsequent figures rarely affect the first six places of the mantissa. Thus, the logarithm of 365.24224 is, to six places of decimals, the same as the logarithm of 365.242.

TO FIND THE NUMBER, ITS LOGARITHM BEING KNOWN.

911. If the mantissa of the logarithm is the same as one in the table, take the corresponding number, and place the decimal point according to the rule of the characteristic.

If the given mantissa is not in the table, find that mantissa in the table which is next less than the given one, and take the corresponding number. Annex to this, two figures of the quotient found by dividing by the tabular difference, the excess of the given mantissa over the one used. Fix the decimal point by the rule of the characteristic.

For example, to find the number whose logarithm is 4.016234.

The next less mantissa is 016197, which has 10380 for its corresponding number (see page 364). The difference between it and the given mantissa is 37, and the tabular difference is 42.

Expressing the fraction $\frac{37}{42}$ decimally, we have the figures 88 to be annexed to those already found, making 1038088, the significant figures of the required number. The characteristic 4 shows that the first significant figure should be in the fifth place. Therefore, 10380.88 is the number sought.

As the logarithms are only approximations, so the number found can only be said to be true for six or seven places of figures. When a greater degree of exactness is required, logarithms must be used of more than six decimal places. These may be calculated by means of Table II, and the formula given with it.

MULTIPLICATION AND DIVISION.

912. Let x and y represent the logarithms of M and N respectively.

By the definition, $10^x = \text{M}$.

Similarly, $10^y = \text{N}$.

Multiplying the first by the second,

$$10^{x+y} = \text{M} \times \text{N}.$$

Dividing the first by the second,

$$10^{x-y} = \text{M} \div \text{N}.$$

That is, $x + y$ is the logarithm of the product of M multiplied by N, and $x - y$ is the logarithm of the quotient of M divided by N. Hence, the following rules for multiplication and division by logarithms:

To multiply, add the logarithms of the factors. The sum is the logarithm of the product.

To divide, subtract the logarithm of the divisor from that of the dividend. The remainder is the logarithm of the quotient.

For example, to find the product of 2, .000314, and 89.235.

$$\begin{aligned} \log.\ 2 &= .301030, \\ \log.\ .000314 &= \bar{4}.496930, \\ \log.\ 89.235 &= 1.950535, \\ \text{The sum,}\quad & \bar{2}.748495 \end{aligned}$$

is the logarithm of .0560396, which is the required product, true to six places of significant figures.

Again, to divide 2 by .000314.

$$\begin{aligned} \log.\ 2 &= .301030, \\ \log.\ .000314 &= \bar{4}.496930, \\ \text{The remainder,}\quad & 3.804100 \end{aligned}$$

is the logarithm of 6369.43, the quotient, true to six places of figures.

Care must be exercised in the additions and subtractions, as the mantissas are all positive and the characteristics sometimes negative.

913. It saves labor, instead of subtracting a logarithm, to add its arithmetical complement. The arithmetical complement is the excess of 10 over the logarithm. Let l represent any logarithm, then $10 - l$ is its complement. If $10 - l$ is added, the result is the same as when l is subtracted and 10 is added. Therefore,

Each time that an arithmetical complement is added, 10 must be subtracted from the result. When the logarithm is itself greater than 10, subtract it from 20 for the complement, and add 20 to the result.

If it were necessary to write out the logarithm in order to subtract it from 10, there would be little saving of labor, but the complement may be written at once, beginning at the left, and subtracting each figure of the given logarithm from 9, to the last significant figure which is to be subtracted from 10. This method is particularly useful when it is required to subtract several logarithms.

For example, to find the value of $\frac{3456 \times 89123}{9753 \times 4321}$.

$$\begin{aligned} \log.\ 3456 &= 3.538574, \\ \log.\ 89123 &= 4.949990, \\ a.\ c.\ \log.\ 9753 &= 6.010862, \\ a.\ c.\ \log.\ 4321 &= \underline{6.364416,} \\ \log.\ 7.30873 &= \ \ .863842. \end{aligned}$$

The sum is diminished by 20, for the complement twice used. Therefore, 7.30873 is the value of the given fraction.

INVOLUTION AND EVOLUTION.

914. Let y represent the logarithm of N. Then,

$$10^y = \mathrm{N}.$$

Raising both members to the x^{th} power,

$$10^{xy} = \mathrm{N}^x.$$

Taking the x^{th} root of both members,

$$10^{\frac{y}{x}} = \sqrt[x]{\mathrm{N}}.$$

That is, xy is the logarithm of the x^{th} power of N, and $\frac{y}{x}$ is the logarithm of the x^{th} root of N. Hence, these rules for involution and evolution by logarithms:

To raise a number to a required power, multiply its logarithm by the exponent of the power. The product is the logarithm of the power.

To extract any root of a number, divide its logarithm by the index of the required root. The quotient is the logarithm of the root.

In making this division, if the characteristic of the given logarithm is negative, and is not exactly divisible by the divisor, then increase it by as many units as are needed to make it so divisible, prefixing the added number to the mantissa as an integer. The result is not affected by thus adding the same number to both the negative and positive parts of the logarithm.

For example, to find the fourth root of $\frac{1}{2}$.

$$\log. .5 = \bar{1}.698970.$$

This logarithm is equal to $-4 + 3.698970$, in which form it may be divided by 4. The quotient $\bar{1}.924742$ is the logarithm of .840896, which is the fourth root of $\frac{1}{2}$.

915. The positive or negative character of a factor is not considered in the use of logarithms. The proper sign can always be given to the result, according to the algebraic principles.

In order that an arithmetical problem may be solved by logarithms, it should not contain any additions or subtractions. If, for example, it is required to find the sum of $\sqrt[3]{3}$ and $\sqrt{2}$, each root may be found separately by the aid of logarithms, but the addition must be made afterward in the usual manner.

Mathematicians have given much attention to the construction of such trigonometrical formulas as require only the operations of multiplication, division, involution, and evolution. For examples of this, see Articles 866 and *seq.* in Plane Triangles, and Articles 880 and *seq.* in Spherical Triangles.

EXERCISES.

916.—1. Calculate the value of these expressions:

$$\sqrt{8932 \times .045726}, \quad \sqrt{7609 \div \sqrt[10]{10}}, \quad \sqrt[12]{13^7 \times 14^8 \div 1.25^6}.$$

2. Find the area of a circle, the radius being 3 feet (500).

3. What is the diameter of a circle whose circumference is 314 feet 3 inches?

4. What is the area of a triangle whose sides are 417, 1493, and 1307 feet? (390.)

5. The diameter of the earth at the equator being 41850000 feet, what is the length in miles of one degree of longitude on the equator, there being 5280 feet in one mile?

6. The earth being a sphere with a radius of 20890000 ft., how many square miles are there in its surface?

Additional exercises may be made upon the formulas of Art. 807.

TABLES

OF

LOGARITHMS OF NUMBERS,

FROM 1 TO 11000,

LOGARITHMS OF 168 PRIME NUMBERS,

TO 15 PLACES OF DECIMALS,

NATURAL SINES AND TANGENTS,

FOR EVERY TEN MINUTES,

AND

LOGARITHMIC SINES AND TANGENTS,

FOR EVERY MINUTE OF THE QUADRANT.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	.0300	.0724	.1147	.1570	.1993	.2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	420
104	7033	7451	7868	8284	8700	9116	9532	9947	.0361	.0775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	.0195	.0600	.1004	.1408	.1812	.2216	.2619	.3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	401
109	7426	7825	8223	8620	9017	9414	9811	.0207	.0602	.0998	397
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
112	9218	9606	9993	.0380	.0766	.1153	.1538	.1924	.2309	.2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	.0320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
117	8186	8557	8928	9298	9668	.0038	.0407	.0776	.1145	.1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	367
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	364
120	079181	9543	9904	.0266	.0626	.0987	.1347	.1707	.2067	.2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	358
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	.0258	.0611	.0963	.1315	.1667	.2018	.2370	.2721	.3071	352
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	096910	7257	7604	7951	8298	8644	8990	9335	9681	.0026	346
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	344
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	.0253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	.0245	330
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	.0012	323
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
138	9879	.0194	.0508	.0822	.1136	.1450	.1763	.2076	.2389	.2702	313
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	9219	9527	9835	.0142	.0449	.0756	.1063	.1370	.1676	.1982	307
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	.0168	.0469	.0769	.1068	301
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	294
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689	288
151	8977	9264	9552	9839	.0126	.0413	.0699	.0986	.1272	.1558	287
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	.0051	281
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	.0029	.0303	.0577	.0850	.1124	274
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
160	204120	4391	4663	4934	5204	5475	5746	6016	6286	6556	270
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	.0051	.0319	.0586	.0853	.1121	.1388	.1654	.1921	267
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	217484	7747	8010	8273	8536	8798	9060	9323	9585	9846	263
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	.0193	256
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	255
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	.0050	.0300	250
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	243038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	.0176	245
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	255273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	267172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	9513	9746	9980	.0213	.0446	.0679	.0912	.1144	.1377	.1609	233
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	278754	8982	9211	9439	9667	9895	.0123	.0351	.0578	.0806	228
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	.0161	.0378	.0595	.0813	218
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	.0056	.0268	.0481	.0693	.0906	.1118	.1330	.1542	212
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	322219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	.0008	.0211	203
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	332438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	.0047	.0246	199
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
220	342423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	.0054	194
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989	194
225	352183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	.0025	.0215	.0404	.0593	.0783	.0972	.1161	.1350	.1539	189
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	.0143	.0328	.0513	.0698	.0883	185
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	.0030	181
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	389166	9343	9520	9698	9875	.0051	.0228	.0405	.0582	.0759	177
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
250	397940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
251	9674	9847	.0020	.0192	.0365	.0538	.0711	.0883	.1056	.1228	173
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	406540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	9933	.0102	.0271	.0440	.0609	.0777	.0946	.1114	.1283	.1451	169
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	414973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
263	9956	.0121	.0286	.0451	.0616	.0781	.0945	.1110	.1275	.1439	165
264	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	423246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
269	9752	9914	.0075	.0236	.0398	.0559	.0720	.0881	.1042	.1203	161
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809	161
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	439333	9491	9648	9806	9964	.0122	.0279	.0437	.0594	.0752	158
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
280	447158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	.0095	154
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	454845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	9392	9543	9694	9845	9995	.0146	.0296	.0447	.0597	.0748	151
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	462398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	469822	9969	.0116	.0263	.0410	.0557	.0704	.0851	.0998	.1145	147
296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
300	477121	7266	7411	7555	7700	7844	7989	8133	8278	8422	145
301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	484300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
309	9958	.0099	.0239	.0380	.0520	.0661	.0801	.0941	.1081	.1222	140
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	498311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
316	9687	9824	9962	.0099	.0236	.0374	.0511	.0648	.0785	.0922	137
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	505150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
323	9203	9337	9471	9606	9740	9874	.0009	.0143	.0277	.0411	134
324	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
325	511883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415	133
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	518514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
331	9828	9959	.0090	.0221	.0353	.0484	.0615	.0745	.0876	.1007	131
332	521138	1269	1400	1530	1661	1792	1922	2053	2183	2314	131
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
335	525045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
338	8917	9045	9174	9302	9430	9559	9687	9815	9943	.0072	128
339	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351	128
340	531479	1607	1734	1862	1990	2117	2245	2372	2500	2627	128
341	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	127
342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	127
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
344	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
345	537819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
346	9076	9202	9327	9452	9578	9703	9829	9954	.0079	.0204	125
347	540329	0455	0580	0705	0830	0955	1080	1205	1330	1454	125
348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
349	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
350	544068	4192	4316	4440	4564	4688	4812	4936	5060	5183	124
351	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
354	9003	9126	9249	9371	9494	9616	9739	9861	9984	.0106	123
355	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	556303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
362	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
363	9907	.0026	.0146	.0265	.0385	.0504	.0624	.0743	.0863	.0982	119
364	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
365	562293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	568202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
371	9374	9491	9608	9725	9842	9959	.0076	.0193	.0309	.0426	117
372	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592	117
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
375	574031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
379	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114
380	579784	9898	.0012	.0126	.0241	.0355	.0469	.0583	.0697	.0811	114
381	580925	1039	1153	1267	1381	1495	1608	1722	1836	1950	114
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	114
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	113
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
385	585461	5574	5686	5799	5912	6024	6137	6250	6362	6475	113
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
388	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	112
389	9950	.0061	.0173	.0284	.0396	.0507	.0619	.0730	.0842	.0953	112
390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066	111
391	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	111
392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	111
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	110
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	110
395	596597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	110
397	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
398	9883	9992	.0101	.0210	.0319	.0428	.0537	.0646	.0755	.0864	109
399	600973	1082	1191	1299	1408	1517	1625	1734	1843	1951	109
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
400	602060	2169	2277	2386	2494	2603	2711	2819	2928	3036	108
401	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	108
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	108
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	108
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	107
405	607455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	107
407	9594	9701	9808	9914	.0021	.0128	.0234	.0341	.0447	.0554	107
408	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617	106
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
410	612784	2890	2996	3102	3207	3313	3419	3525	3630	3736	106
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	106
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	105
415	618048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	.0032	104
417	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072	104
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
420	623249	3353	3456	3559	3663	3766	3869	3973	4076	4179	103
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	103
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	103
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
425	628389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
426	9410	9512	9613	9715	9817	9919	.0021	.0123	.0224	.0326	102
427	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342	102
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
430	633468	3569	3670	3771	3872	3973	4074	4175	4276	4376	101
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	101
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	100
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	100
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	100
435	638489	8589	8689	8789	8888	8988	9088	9188	9287	9387	100
436	9486	9586	9686	9785	9885	9984	.0084	.0183	.0283	.0382	99
437	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375	99
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	99
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
440	643453	3551	3650	3749	3847	3946	4044	4143	4242	4340	99
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	98
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	98
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
445	648360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
446	9335	9432	9530	9627	9724	9821	9919	.0016	.0113	.0210	97
447	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181	97
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
450	653213	3309	3405	3502	3598	3695	3791	3888	3984	4080	96
451	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	96
452	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	96
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
455	658011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
457	9916	.0011	.0106	.0201	.0296	.0391	.0486	.0581	.0676	.0771	95
458	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718	95
459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	95
460	662758	2852	2947	3041	3135	3230	3324	3418	3512	3607	94
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	94
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	94
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	94
465	667453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
467	9317	9410	9503	9596	9689	9782	9875	9967	.0060	.0153	93
468	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080	93
469	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	93
470	672098	2190	2283	2375	2467	2560	2652	2744	2836	2929	92
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	92
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
475	676694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
478	9428	9519	9610	9700	9791	9882	9973	.0063	.0154	.0245	91
479	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151	91
480	681241	1332	1422	1513	1603	1693	1784	1874	1964	2055	90
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	90
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	90
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	90
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
485	685742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
489	9309	9398	9486	9575	9664	9753	9841	9930	.0019	.0107	89
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993	89
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
495	694605	4693	4781	4868	4956	5044	5131	5219	5307	5394	88
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
500	698970	9057	9144	9231	9317	9404	9491	9578	9664	9751	87
501	9838	9924	.0011	.0098	.0184	.0271	.0358	.0444	.0531	.0617	87
502	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482	86
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	86
505	703291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
510	707570	7655	7740	7826	7911	7996	8081	8166	8251	8336	85
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	.0033	85
513	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879	85
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
515	711807	1892	1976	2060	2144	2229	2313	2397	2481	2566	84
516	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	84
520	716003	6087	6170	6254	6337	6421	6504	6588	6671	6754	83
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	83
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	.0077	83
525	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903	83
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
530	724276	4358	4440	4522	4604	4685	4767	4849	4931	5013	82
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
535	728354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
537	9974	.0055	.0136	.0217	.0298	.0378	.0459	.0540	.0621	.0702	81
538	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
540	732394	2474	2555	2635	2715	2796	2876	2956	3037	3117	80
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	80
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	80
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	80
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
545	736397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
549	9572	9651	9731	9810	9889	9968	.0047	.0126	.0205	.0284	79
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
550	740363	0442	0521	0600	0678	0757	0836	0915	0994	1073	79
551	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
555	744293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
560	748188	8266	8343	8421	8498	8576	8653	8731	8808	8885	77
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
562	9736	9814	9891	9968	.0045	.0123	.0200	.0277	.0354	.0431	77
563	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202	77
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
565	752048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
570	755875	5951	6027	6103	6180	6256	6332	6408	6484	6560	76
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
575	759668	9743	9819	9894	9970	.0045	.0121	.0196	.0272	.0347	75
576	760422	0498	0573	0649	0724	0799	0875	0950	1025	1101	75
577	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
579	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	75
580	763428	3503	3578	3653	3727	3802	3877	3952	4027	4101	75
581	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	75
582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
585	767156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
588	9377	9451	9525	9599	9673	9746	9820	9894	9968	.0042	74
589	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778	74
590	770852	0926	0999	1073	1146	1220	1293	1367	1440	1514	74
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	774517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
599	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
N.	0	1	2	3	4	5	6	7	8	9	D.

TABLE I.—LOGARITHMS

N.	0	1	2	3	4	5	6	7	8	9	D.
600	778151	8224	8296	8368	8441	8513	8585	8658	8730	8802	72
601	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	72
602	9596	9669	9741	9813	9885	9957	.0029	.0101	.0173	.0245	72
603	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965	72
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
605	781755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
610	785330	5401	5472	5543	5615	5686	5757	5828	5899	5970	71
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
615	788875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
616	9581	9651	9722	9792	9863	9933	.0004	.0074	.0144	.0215	70
617	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918	70
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
620	792392	2462	2532	2602	2672	2742	2812	2882	2952	3022	70
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	70
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
625	795880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	799341	9409	9478	9547	9616	9685	9754	9823	9892	9961	69
631	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648	69
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	69
635	802774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
640	806180	6248	6316	6384	6451	6519	6587	6655	6723	6790	68
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
645	809560	9627	9694	9762	9829	9896	9964	.0031	.0098	.0165	67
646	810233	0300	0367	0434	0501	0569	0636	0703	0770	0837	67
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
650	812913	2980	3047	3114	3181	3247	3314	3381	3448	3514	67
651	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
655	816241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
660	819544	9610	9676	9741	9807	9873	9939	.0004	.0070	.0136	66
661	820201	0267	0333	0399	0464	0530	0595	0661	0727	0792	66
662	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	66
663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
664	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
665	822822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
670	826075	6140	6204	6269	6334	6399	6464	6528	6593	6658	65
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
675	829304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
676	9947	.0011	.0075	.0139	.0204	.0268	.0332	.0396	.0460	.0525	64
677	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
680	832509	2573	2637	2700	2764	2828	2892	2956	3020	3083	64
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	64
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
685	835691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	838849	8912	8975	9038	9101	9164	9227	9289	9352	9415	63
691	9478	9541	9604	9667	9729	9792	9855	9918	9981	.0043	63
692	840106	0169	0232	0294	0357	0420	0482	0545	0608	0671	63
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
695	841985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
699	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	62
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
700	845098	5160	5222	5284	5346	5408	5470	5532	5594	5656	62
701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
705	848189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
708	850033	0095	0156	0217	0279	0340	0401	0462	0524	0585	61
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
710	851258	1320	1381	1442	1503	1564	1625	1686	1747	1809	61
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
715	854306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
720	857332	7393	7453	7513	7574	7634	7694	7755	7815	7875	60
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
724	9739	9799	9859	9918	9978	.0038	.0098	.0158	.0218	.0278	60
725	860338	0398	0458	0518	0578	0637	0697	0757	0817	0877	60
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
730	863323	3382	3442	3501	3561	3620	3680	3739	3799	3858	59
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
735	866287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
740	869232	9290	9349	9408	9466	9525	9584	9642	9701	9760	59
741	9818	9877	9935	9994	.0053	.0111	.0170	.0228	.0287	.0345	59
742	870404	0462	0521	0579	0638	0696	0755	0813	0872	0930	58
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
745	872156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
750	875061	5119	5177	5235	5293	5351	5409	5466	5524	5582	58
751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
755	877947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
758	9669	9726	9784	9841	9898	9956	.0013	.0070	.0127	.0185	57
759	880242	0299	0356	0413	0471	0528	0585	0642	0699	0756	57
760	880814	0871	0928	0985	1042	1099	1156	1213	1271	1328	57
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
765	883661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
770	886491	6547	6604	6660	6716	6773	6829	6885	6942	6998	56
771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
775	889302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
776	9862	9918	9974	.0030	.0086	.0141	.0197	.0253	.0309	.0365	56
777	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924	56
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
780	892095	2150	2206	2262	2317	2373	2429	2484	2540	2595	56
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
785	894870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	897627	7682	7737	7792	7847	7902	7957	8012	8067	8122	55
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
794	9821	9875	9930	9985	.0039	.0094	.0149	.0203	.0258	.0312	55
795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859	55
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
800	903090	3144	3199	3253	3307	3361	3416	3470	3524	3578	54
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	905796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
810	908485	8539	8592	8646	8699	8753	8807	8860	8914	8967	54
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	.0037	53
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571	53
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
815	911158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
820	913814	3867	3920	3973	4026	4079	4132	4184	4237	4290	53
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
825	916454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	919078	9130	9183	9235	9287	9340	9392	9444	9496	9549	52
831	9601	9653	9706	9758	9810	9862	9914	9967	.0019	.0071	52
832	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593	52
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	921686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	924279	4331	4383	4434	4486	4538	4589	4641	4693	4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	926857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
850	929419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51
851	9930	9981	.0032	.0083	.0134	.0185	.0236	.0287	.0338	.0389	51
852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898	51
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	931966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
860	934498	4549	4599	4650	4700	4751	4801	4852	4902	4953	50
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
865	937016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	939519	9569	9619	9669	9719	9769	9819	9869	9918	9968	50
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467	50
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
875	942008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
880	944483	4532	4581	4631	4680	4729	4779	4828	4877	4927	49
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	946943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	949390	9439	9488	9536	9585	9634	9683	9731	9780	9829	49
891	9878	9926	9975	.0024	.0073	.0121	.0170	.0219	.0267	.0316	49
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803	49
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
895	951823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
900	954243	4291	4339	4387	4435	4484	4532	4580	4628	4677	48
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	956649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	959041	9089	9137	9185	9232	9280	9328	9375	9423	9471	48
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
912	9995	.0042	.0090	.0138	.0185	.0233	.0280	.0328	.0376	.0423	48
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	47
915	961421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	963788	3835	3882	3929	3977	4024	4071	4118	4165	4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	966142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	968483	8530	8576	8623	8670	8716	8763	8810	8856	8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	9882	9928	9975	.0021	.0068	.0114	.0161	.0207	.0254	.0300	47
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765	46
935	970812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
940	973128	3174	3220	3266	3313	3359	3405	3451	3497	3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	975432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	982271	2316	2362	2407	2452	2497	2543	2588	2633	2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	984527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	986772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	989005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	.0028	.0072	.0117	.0161	.0206	.0250	.0294	44
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738	44
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	991226	1270	1315	1359	1403	1448	1492	1536	1580	1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	993436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	997823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	43
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
1000	000000	0043	0087	0130	0174	0217	0260	0304	0347	0391	43
1001	0434	0477	0521	0564	0608	0651	0694	0738	0781	0824	43
1002	0868	0911	0954	0998	1041	1084	1128	1171	1214	1258	43
1003	1301	1344	1388	1431	1474	1517	1561	1604	1647	1690	43
1004	1734	1777	1820	1863	1907	1950	1993	2036	2080	2123	43
1005	002166	2209	2252	2296	2339	2382	2425	2468	2512	2555	43
1006	2598	2641	2684	2727	2771	2814	2857	2900	2943	2986	43
1007	3029	3073	3116	3159	3202	3245	3288	3331	3374	3417	43
1008	3461	3504	3547	3590	3633	3676	3719	3762	3805	3848	43
1009	3891	3934	3977	4020	4063	4106	4149	4192	4235	4278	43
1010	004321	4364	4407	4450	4493	4536	4579	4622	4665	4708	43
1011	4751	4794	4837	4880	4923	4966	5009	5052	5095	5138	43
1012	5181	5223	5266	5309	5352	5395	5438	5481	5524	5567	43
1013	5609	5652	5695	5738	5781	5824	5867	5909	5952	5995	43
1014	6038	6081	6124	6166	6209	6252	6295	6338	6380	6423	43
1015	006466	6509	6552	6594	6637	6680	6723	6765	6808	6851	43
1016	6894	6936	6979	7022	7065	7107	7150	7193	7236	7278	43
1017	7321	7364	7406	7449	7492	7534	7577	7620	7662	7705	43
1018	7748	7790	7833	7876	7918	7961	8004	8046	8089	8132	43
1019	8174	8217	8259	8302	8345	8387	8430	8472	8515	8558	43
1020	008600	8643	8685	8728	8770	8813	8856	8898	8941	8983	43
1021	9026	9068	9111	9153	9196	9238	9281	9323	9366	9408	42
1022	9451	9493	9536	9578	9621	9663	9706	9748	9791	9833	42
1023	9876	9918	9961	.0003	.0045	.0088	.0130	.0173	.0215	.0258	42
1024	010300	0342	0385	0427	0470	0512	0554	0597	0639	0681	42
1025	010724	0766	0809	0851	0893	0936	0978	1020	1063	1105	42
1026	1147	1190	1232	1274	1317	1359	1401	1444	1486	1528	42
1027	1570	1613	1655	1697	1740	1782	1824	1866	1909	1951	42
1028	1993	2035	2078	2120	2162	2204	2247	2289	2331	2373	42
1029	2415	2458	2500	2542	2584	2626	2669	2711	2753	2795	42
1030	012837	2879	2922	2964	3006	3048	3090	3132	3174	3217	42
1031	3259	3301	3343	3385	3427	3469	3511	3553	3596	3638	42
1032	3680	3722	3764	3806	3848	3890	3932	3974	4016	4058	42
1033	4100	4142	4184	4226	4268	4310	4353	4395	4437	4479	42
1034	4521	4563	4605	4647	4689	4730	4772	4814	4856	4898	42
1035	014940	4982	5024	5066	5108	5150	5192	5234	5276	5318	42
1036	5360	5402	5444	5485	5527	5569	5611	5653	5695	5737	42
1037	5779	5821	5863	5904	5946	5988	6030	6072	6114	6156	42
1038	6197	6239	6281	6323	6365	6407	6448	6490	6532	6574	42
1039	6616	6657	6699	6741	6783	6824	6866	6908	6950	6992	42
1040	017033	7075	7117	7159	7200	7242	7284	7326	7367	7409	42
1041	7451	7492	7534	7576	7618	7659	7701	7743	7784	7826	42
1042	7868	7909	7951	7993	8034	8076	8118	8159	8201	8243	42
1043	8284	8326	8368	8409	8451	8492	8534	8576	8617	8659	42
1044	8700	8742	8784	8825	8867	8908	8950	8992	9033	9075	42
1045	019116	9158	9199	9241	9282	9324	9366	9407	9449	9490	42
1046	9532	9573	9615	9656	9698	9739	9781	9822	9864	9905	42
1047	9947	9988	.0030	.0071	.0113	.0154	.0195	.0237	.0278	.0320	41
1048	020361	0403	0444	0486	0527	0568	0610	0651	0693	0734	41
1049	0775	0817	0858	0900	0941	0982	1024	1065	1107	1148	41
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
1050	021189	1231	1272	1313	1355	1396	1437	1479	1520	1561	41
1051	1603	1644	1685	1727	1768	1809	1851	1892	1933	1974	41
1052	2016	2057	2098	2140	2181	2222	2263	2305	2346	2387	41
1053	2428	2470	2511	2552	2593	2635	2676	2717	2758	2799	41
1054	2841	2882	2923	2964	3005	3047	3088	3129	3170	3211	41
1055	023252	3294	3335	3376	3417	3458	3499	3541	3582	3623	41
1056	3664	3705	3746	3787	3828	3870	3911	3952	3993	4034	41
1057	4075	4116	4157	4198	4239	4280	4321	4363	4404	4445	41
1058	4486	4527	4568	4609	4650	4691	4732	4773	4814	4855	41
1059	4896	4937	4978	5019	5060	5101	5142	5183	5224	5265	41
1060	025306	5347	5388	5429	5470	5511	5552	5593	5634	5674	41
1061	5715	5756	5797	5838	5879	5920	5961	6002	6043	6084	41
1062	6125	6165	6206	6247	6288	6329	6370	6411	6452	6492	41
1063	6533	6574	6615	6656	6697	6737	6778	6819	6860	6901	41
1064	6942	6982	7023	7064	7105	7146	7186	7227	7268	7309	41
1065	027350	7390	7431	7472	7513	7553	7594	7635	7676	7716	41
1066	7757	7798	7839	7879	7920	7961	8002	8042	8083	8124	41
1067	8164	8205	8246	8287	8327	8368	8409	8449	8490	8531	41
1068	8571	8612	8653	8693	8734	8775	8815	8856	8896	8937	41
1069	8978	9018	9059	9100	9140	9181	9221	9262	9303	9343	41
1070	029384	9424	9465	9506	9546	9587	9627	9668	9708	9749	41
1071	9789	9830	9871	9911	9952	9992	.0033	.0073	.0114	.0154	41
1072	030195	0235	0276	0316	0357	0397	0438	0478	0519	0559	40
1073	0600	0640	0681	0721	0762	0802	0843	0883	0923	0964	40
1074	1004	1045	1085	1126	1166	1206	1247	1287	1328	1368	40
1075	031408	1449	1489	1530	1570	1610	1651	1691	1732	1772	40
1076	1812	1853	1893	1933	1974	2014	2054	2095	2135	2175	40
1077	2216	2256	2296	2337	2377	2417	2458	2498	2538	2578	40
1078	2619	2659	2699	2740	2780	2820	2860	2901	2941	2981	40
1079	3021	3062	3102	3142	3182	3223	3263	3303	3343	3384	40
1080	033424	3464	3504	3544	3585	3625	3665	3705	3745	3786	40
1081	3826	3866	3906	3946	3986	4027	4067	4107	4147	4187	40
1082	4227	4267	4308	4348	4388	4428	4468	4508	4548	4588	40
1083	4628	4669	4709	4749	4789	4829	4869	4909	4949	4989	40
1084	5029	5069	5109	5149	5190	5230	5270	5310	5350	5390	40
1085	035430	5470	5510	5550	5590	5630	5670	5710	5750	5790	40
1086	5830	5870	5910	5950	5990	6030	6070	6110	6150	6190	40
1087	6230	6269	6309	6349	6389	6429	6469	6509	6549	6589	40
1088	6629	6669	6709	6749	6789	6828	6868	6908	6948	6988	40
1089	7028	7068	7108	7148	7187	7227	7267	7307	7347	7387	40
1090	037426	7466	7506	7546	7586	7626	7665	7705	7745	7785	40
1091	7825	7865	7904	7944	7984	8024	8064	8103	8143	8183	40
1092	8223	8262	8302	8342	8382	8421	8461	8501	8541	8580	40
1093	8620	8660	8700	8739	8779	8819	8859	8898	8938	8978	40
1094	9017	9057	9097	9136	9176	9216	9255	9295	9335	9374	40
1095	039414	9454	9493	9533	9573	9612	9652	9692	9731	9771	40
1096	9811	9850	9890	9929	9969	.0009	.0048	.0088	.0127	.0167	40
1097	040207	0246	0286	0325	0365	0405	0444	0484	0523	0563	40
1098	0602	0642	0681	0721	0761	0800	0840	0879	0919	0958	40
1099	0998	1037	1077	1116	1156	1195	1235	1274	1314	1353	39
N.	0	1	2	3	4	5	6	7	8	9	D.

TABLE II.—LOGARITHMS OF PRIME

N.	Logarithm.	N.	Logarithm.	N.	Logarithm.
2	30102 99956 63981	233	36735 59210 26019	547	73798 73263 33431
3	47712 12547 19662	239	37839 79009 48138	557	74585 51951 73729
5	69897 00043 36019	241	38201 70425 74868	563	75050 83948 51346
7	84509 80400 14257	251	39967 37214 81038	569	75511 22663 95071
11	04139 26851 58225	257	40993 31233 31295	571	75663 61082 45848
13	11394 33523 06837	263	41995 57484 89758	577	76117 58131 55731
17	23044 89213 78274	269	42975 22800 02408	587	76863 81012 47614
19	27875 36009 52829	271	43296 92908 74406	593	77305 46933 64263
23	36172 78360 17593	277	44247 97690 64449	599	77742 68223 89311
29	46239 79978 98956	281	44870 63199 05080	601	77887 44720 02740
31	49136 16938 34273	283	45178 64355 24290	607	78318 86910 75258
37	56820 17240 66995	293	46686 76203 54109	613	78746 04745 18415
41	61278 38567 19735	307	48713 83754 77186	617	79028 51640 33242
43	63346 84555 79587	311	49276 03890 26838	619	79169 06490 20118
47	67209 78579 35717	313	49554 43375 46448	631	80002 93592 44134
53	72427 58696 00789	317	50105 92622 17751	641	80685 80295 18817
59	77085 20116 42144	331	51982 79937 75719	643	80821 09729 24222
61	78532 98350 10767	337	52762 99008 71339	647	81090 42806 68700
67	82607 48027 00826	347	54032 94747 90874	653	81491 31812 75074
71	85125 83487 19075	349	54282 54269 59180	659	81888 54145 94010
73	86332 28601 20456	353	54777 47053 87823	661	82020 14594 85640
79	89762 70912 90441	359	55509 44485 78319	673	82801 50642 23977
83	91907 80923 76074	367	56466 60642 52089	677	83058 86686 85144
89	94939 00066 44913	373	57170 88318 08688	683	83442 07036 81533
97	98677 17342 66245	379	57863 92099 68072	691	83947 80473 74198
101	00432 13737 82643	383	58319 87739 68623	701	84571 80179 66659
103	01283 72247 05172	389	58994 96013 25708	709	85064 62351 83067
107	02938 37776 85210	397	59879 05067 63115	719	85672 88903 82883
109	03742 64979 40624	401	60314 43726 20182	727	86153 44108 59038
113	05307 84434 83420	409	61172 33080 07342	733	86510 39746 41128
127	10380 37209 55957	419	62221 40229 66295	739	86864 44383 94826
131	11727 12956 55764	421	62428 20958 35668	743	87098 88137 60575
137	13672 05671 56407	431	63447 72701 60732	751	87563 99370 04168
139	14301 48002 54095	433	63648 78963 53365	757	87909 58795 00073
149	17318 62684 12274	439	64246 45202 42121	761	88138 46567 70573
151	17897 69472 93169	443	64640 37262 23070	769	88592 63398 01431
157	19589 96524 09234	449	65224 63410 03323	773	88817 94939 18325
163	21218 76044 03958	457	65991 62000 69850	787	89597 47323 59065
167	22271 64711 47583	461	66370 09253 89648	797	90145 83213 96112
173	23804 61031 28795	463	66558 09910 17953	809	90794 85216 12272
179	25285 30309 79893	467	66931 68805 66112	811	90902 08542 11156
181	25767 85748 69185	479	68033 55134 14563	821	91434 31571 19441
191	28103 33672 47728	487	68752 89612 14634	823	91539 98352 12270
193	28555 73090 07774	491	69108 14921 22968	827	91750 55095 52547
197	29446 62261 61593	499	69810 05456 23390	829	91855 45305 50274
199	29885 30764 09707	503	70156 79850 55927	839	92376 19608 28700
211	32428 24552 97693	509	70671 77823 36759	853	93094 90311 67523
223	34830 48630 48161	521	71683 77232 99524	857	93298 08219 23198
227	35602 58571 93123	523	71850 16888 67274	859	93399 31638 31242
229	35983 54823 39888	541	73319 72651 06569	863	93601 07957 15210

NUMBERS LESS THAN 1000.

N.	Logarithm.	N.	Logarithm.	N.	Logarithm.
877	94299 95933 66041	919	96331 55113 86111	967	98542 64740 83002
881	94497 59084 12048	929	96801 57139 93642	971	98721 92299 08005
883	94596 07035 77569	937	97173 95908 87778	977	98989 45637 18773
887	94792 36198 31726	941	97358 96234 27257	983	99255 35178 32136
907	95760 72870 60095	947	97634 99790 03273	991	99607 36544 85275
911	95951 83769 72998	953	97909 29006 38326	997	99869 51583 11656

In the above table, only the mantissas are given; the characteristics may be found by the rule (908).

By means of these logarithms, the logarithm of any number may be found with equal accuracy. If the given number be the product of any of the prime numbers in the table, its logarithm may be found by addition (912). For example,

$$\text{log. } 6 = \text{log. } 2 + \text{log. } 3 = \;.77815\ 12503\ 83643;$$
$$\text{log. } 1001 = \text{log. } 7 + \text{log. } 11 + \text{log. } 13 = 3.00043\ 40774\ 79319.$$

These results may err in the last figure; the logarithm of 6 to fifteen figures, has the last figure nearer to 4 than to 3.

When the given number is not the product of numbers in the table, its logarithm may be calculated by the following formulas:

$$M = .43429\ 44819\ 0325;$$
$$\text{log. } n = \text{log. } (n-1) + 2\,M\left(\frac{1}{2\,n-1} + \frac{1}{3\,(2\,n-1)^3} + \&c\right).$$

Omitting the second fraction in the parenthesis, the logarithm will be found correct to three times as many figures as there are in the number *n*. Using this term gives the result true to five times as many figures as there are in *n*. For example, to find the logarithm of 1013,

$$\begin{aligned} \text{log. } 1012 = 2\text{ log. } 2 + \text{log. } 11 + \text{log. } 23 &= 3.00518\ 05125\ 03780 \\ 2\,M \div 2025 &= \;.00042\ 89328\ 21633 \\ 2\,M \div 3(2025)^3 &= \;.00000\ 00000\ 34867 \\ \hline \text{log. } 1013 &= 3.00560\ 94453\ 60280 \end{aligned}$$

For some large numbers it may be necessary to repeat the operation. When one of the prime factors of $n-1$ is greater than 1000, it may be better to find the logarithm of $n+1$, and then log. *n* by *subtracting* the difference. For example, log. 2027 can be found more readily from log. 2028 than from log. 2026.

TABLE III.—NATURAL SINES.

Deg.	0′	10′	20′	30′	40′	50′	60′	Deg.
0	000000	002909	005818	008727	011635	014544	017452	89
1	017452	020361	023269	026177	029085	031992	034899	88
2	034899	037806	040713	043619	046525	049431	052336	87
3	052336	055241	058145	061049	063952	066854	069756	86
4	069756	072658	075559	078459	081359	084258	087156	85
5	087156	090053	092950	095846	098741	101635	104528	84
6	104528	107421	110313	113203	116093	118982	121869	83
7	121869	124756	127642	130526	133410	136292	139173	82
8	139173	142053	144932	147809	150686	153561	156434	81
9	156434	159307	162178	165048	167916	170783	173648	80
10	173648	176512	179375	182236	185095	187953	190809	79
11	190809	193664	196517	199368	202218	205065	207912	78
12	207912	210756	213599	216440	219279	222116	224951	77
13	224951	227784	230616	233445	236273	239098	241922	76
14	241922	244743	247563	250380	253195	256008	258819	75
15	258819	261628	264434	267238	270040	272840	275637	74
16	275637	278432	281225	284015	286803	289589	292372	73
17	292372	295152	297930	300706	303479	306249	309017	72
18	309017	311782	314545	317305	320062	322816	325568	71
19	325568	328317	331063	333807	336547	339285	342020	70
20	342020	344752	347481	350207	352931	355651	358368	69
21	358368	361082	363793	366501	369206	371908	374607	68
22	374607	377302	379994	382683	385369	388052	390731	67
23	390731	393407	396080	398749	401415	404078	406737	66
24	406737	409392	412045	414693	417338	419980	422618	65
25	422618	425253	427884	430511	433135	435755	438371	64
26	438371	440984	443593	446198	448799	451397	453990	63
27	453990	456580	459166	461749	464327	466901	469472	62
28	469472	472038	474600	477159	479713	482263	484810	61
29	484810	487352	489890	492424	494953	497479	500000	60
30	500000	502517	505030	507538	510043	512543	515038	59
31	515038	517529	520016	522499	524977	527450	529919	58
32	529919	532384	534844	537300	539751	542197	544639	57
33	544639	547076	549509	551937	554360	556779	559193	56
34	559193	561602	564007	566406	568801	571191	573576	55
35	573576	575957	578332	580703	583069	585429	587785	54
36	587785	590136	592482	594823	597159	599489	601815	53
37	601815	604136	606451	608761	611067	613367	615661	52
38	615661	617951	620235	622515	624789	627057	629320	51
39	629320	631578	633831	636078	638320	640557	642788	50
40	642788	645013	647233	649448	651657	653861	656059	49
41	656059	658252	660439	662620	664796	666966	669131	48
42	669131	671289	673443	675590	677732	679868	681998	47
43	681998	684123	686242	688355	690462	692563	694658	46
44	694658	696748	698832	700909	702981	705047	707107	45
Deg.	60′	50′	40′	30′	20′	10′	0′	Deg.

NATURAL COSINES.

TABLE III.—NATURAL TANGENTS.

Deg.	0′	10′	20′	30′	40′	50′	60′	Deg.
0	000000	002909	005818	008727	011636	014545	017455	89
1	017455	020365	023275	026186	029097	032009	034921	88
2	034921	037834	040747	043661	046576	049491	052408	87
3	052408	055325	058243	061163	064083	067004	069927	86
4	069927	072851	075775	078702	081629	084558	087489	85
5	087489	090421	093354	096289	099226	102164	105104	84
6	105104	108046	110990	113936	116883	119833	122785	83
7	122785	125738	128694	131652	134613	137576	140541	82
8	140541	143508	146478	149451	152426	155404	158384	81
9	158384	161368	164354	167343	170334	173329	176327	80
10	176327	179328	182332	185339	188349	191363	194380	79
11	194380	197401	200425	203452	206483	209518	212557	78
12	212557	215599	218645	221695	224748	227806	230868	77
13	230868	233934	237004	240079	243157	246241	249328	76
14	249328	252420	255516	258618	261723	264834	267949	75
15	267949	271069	274194	277325	280460	283600	286745	74
16	286745	289896	293052	296213	299380	302553	305731	73
17	305731	308914	312104	315299	318500	321707	324920	72
18	324920	328139	331364	334595	337833	341077	344328	71
19	344328	347585	350848	354119	357396	360679	363970	70
20	363970	367268	370573	373885	377204	380530	383864	69
21	383864	387205	390554	393910	397275	400646	404026	68
22	404026	407414	410810	414214	417626	421046	424475	67
23	424475	427912	431358	434812	438276	441748	445229	66
24	445229	448719	452218	455726	459244	462771	466308	65
25	466308	469854	473410	476976	480551	484137	487733	64
26	487733	491339	494955	498582	502219	505867	509525	63
27	509525	513195	516875	520567	524270	527984	531709	62
28	531709	535446	539195	542956	546728	550513	554309	61
29	554309	558118	561939	565773	569619	573478	577350	60
30	577350	581235	585134	589045	592970	596908	600861	59
31	600861	604827	608807	612801	616809	620832	624869	58
32	624869	628921	632988	637070	641167	645280	649408	57
33	649408	653551	657710	661886	666077	670284	674509	56
34	674509	678749	683007	687281	691572	695881	700208	55
35	700208	704551	708913	713293	717691	722108	726543	54
36	726543	730996	735469	739961	744472	749003	753554	53
37	753554	758125	762716	767327	771959	776612	781286	52
38	781286	785981	790697	795436	800196	804979	809784	51
39	809784	814612	819463	824336	829234	834155	839100	50
40	839100	844069	849062	854081	859124	864193	869287	49
41	869287	874407	879553	884725	889924	895151	900404	48
42	900404	905685	910994	916331	921697	927091	932515	47
43	932515	937968	943451	948965	954508	960083	965689	46
44	965689	971326	976996	982697	988432	994199	1.000000	45
Deg.	60′	50′	40′	30′	20′	10′	0′	Deg.

NATURAL COTANGENTS.

TABLE III.—NATURAL SINES.

Deg.	0′	10′	20′	30′	40′	50′	60′	Deg.
45	707107	709161	711209	713250	715286	717316	719340	44
46	719340	721357	723369	725374	727374	729367	731354	43
47	731354	733334	735309	737277	739239	741195	743145	42
48	743145	745088	747025	748956	750880	752798	754710	41
49	754710	756615	758511	760406	762292	764171	766044	40
50	766044	767911	769771	771625	773472	775312	777146	39
51	777146	778973	780794	782608	784416	786217	788011	38
52	788011	789798	791579	793353	795121	796882	798636	37
53	798636	800383	802123	803857	805584	807304	809017	36
54	809017	810723	812423	814116	815801	817480	819152	35
55	819152	820817	822475	824126	825770	827407	829038	34
56	829038	830661	832277	833886	835488	837083	838671	33
57	838671	840251	841825	843391	844951	846503	848048	32
58	848048	849586	851117	852640	854156	855665	857167	31
59	857167	858662	860149	861629	863102	864567	866025	30
60	866025	867476	868920	870356	871784	873206	874620	29
61	874620	876026	877425	878817	880201	881578	882948	28
62	882948	884309	885664	887011	888350	889682	891007	27
63	891007	892323	893633	894934	896229	897515	898794	26
64	898794	900065	901329	902585	903834	905075	906308	25
65	906308	907533	908751	909961	911164	912358	913545	24
66	913545	914725	915896	917060	918216	919364	920505	23
67	920505	921638	922762	923880	924989	926090	927184	22
68	927184	928270	929348	930418	931480	932534	933580	21
69	933580	934619	935650	936672	937687	938694	939693	20
70	939693	940684	941666	942641	943609	944568	945519	19
71	945519	946462	947397	948324	949243	950154	951057	18
72	951057	951951	952838	953717	954588	955450	956305	17
73	956305	957151	957990	958820	959642	960456	961262	16
74	961262	962059	962849	963630	964404	965169	965926	15
75	965926	966675	967415	968148	968872	969588	970296	14
76	970296	970995	971687	972370	973045	973712	974370	13
77	974370	975020	975662	976296	976921	977539	978148	12
78	978148	978748	979341	979925	980500	981068	981627	11
79	981627	982178	982721	983255	983781	984298	984808	10
80	984808	985309	985801	986286	986762	987229	987688	9
81	987688	988139	988582	989016	989442	989859	990268	8
82	990268	990669	991061	991445	991820	992187	992546	7
83	992546	992896	993238	993572	993897	994214	994522	6
84	994522	994822	995113	995396	995671	995937	996195	5
85	996195	996444	996685	996917	997141	997357	997564	4
86	997564	997763	997953	998135	998308	998473	998630	3
87	998630	998778	998917	999048	999171	999285	999391	2
88	999391	999488	999577	999657	999729	999793	999848	1
89	999848	999894	999932	999962	999983	999996	1.000000	0
Deg.	60′	50′	40′	30′	20′	10′	0′	Deg.

NATURAL COSINES.

TABLE III.—NATURAL TANGENTS.

Deg.	0′	10′	20′	30′	40′	50′	60′	Deg.
45	1.000000	1.005835	1.011704	1.017607	1.023546	1.029520	1.035530	44
46	1.035530	1.041577	1.047660	1.053780	1.059938	1.066134	1.072369	43
47	1.072369	1.078642	1.084955	1.091309	1.097702	1.104137	1.110612	42
48	1.110612	1.117131	1.123691	1.130294	1.136941	1.143633	1.150368	41
49	1.150368	1.157149	1.163976	1.170850	1.177770	1.184738	1.191754	40
50	1.191754	1.198818	1.205933	1.213097	1.220312	1.227579	1.234897	39
51	1.234897	1.242269	1.249693	1.257172	1.264706	1.272296	1.279942	38
52	1.279942	1.287645	1.295406	1.303225	1.311105	1.319044	1.327045	37
53	1.327045	1.335108	1.343233	1.351422	1.359676	1.367996	1.376382	36
54	1.376382	1.384835	1.393357	1.401948	1.410610	1.419343	1.428148	35
55	1.428148	1.437027	1.445980	1.455009	1.464115	1.473298	1.482561	34
56	1.482561	1.491904	1.501328	1.510835	1.520426	1.530102	1.539865	33
57	1.539865	1.549716	1.559655	1.569686	1.579808	1.590024	1.600335	32
58	1.600335	1.610742	1.621247	1.631852	1.642558	1.653366	1.664279	31
59	1.664279	1.675299	1.686426	1.697663	1.709012	1.720474	1.732051	30
60	1.732051	1.743745	1.755559	1.767494	1.779552	1.791736	1.804048	29
61	1.804048	1.816489	1.829063	1.841771	1.854616	1.867600	1.880726	28
62	1.880726	1.893997	1.907415	1.920982	1.934702	1.948577	1.962611	27
63	1.962311	1.976805	1.991164	2.005690	2.020386	2.035256	2.050304	26
64	2.050304	2.065532	2.080944	2.096544	2.112335	2.128321	2.144507	25
65	2.144507	2.160896	2.177492	2.194300	2.211323	2.228568	2.246037	24
66	2.246037	2.263736	2.281669	2.299843	2.318261	2.336929	2.355852	23
67	2.355852	2.375037	2.394489	2.414214	2.434217	2.454506	2.475087	22
68	2.475087	2.495966	2.517151	2.538648	2.560465	2.582609	2.605089	21
69	2.605089	2.627912	2.651087	2.674621	2.698525	2.722808	2.747477	20
70	2.747477	2.772545	2.798020	2.823913	2.850235	2.876997	2.904211	19
71	2.904211	2.931888	2.960042	2.988685	3.017830	3.047492	3.077684	18
72	3.077684	3.108421	3.139719	3.171595	3.204064	3.237144	3.270853	17
73	3.270853	3.305209	3.340233	3.375943	3.412363	3.449512	3.487414	16
74	3.487414	3.526094	3.565575	3.605884	3.647047	3.689093	3.732051	15
75	3.732051	3.775952	3.820828	3.866713	3.913642	3.961652	4.010781	14
76	4.010781	4.061070	4.112561	4.165300	4.219332	4.274707	4.331476	13
77	4.331476	4.389694	4.449418	4.510709	4.573629	4.638246	4.704630	12
78	4.704630	4.772857	4.843005	4.915157	4.989403	5.065835	5.144554	11
79	5.144554	5.225665	5.309279	5.395517	5.484505	5.576379	5.671282	10
80	5.671282	5.769369	5.870804	5.975764	6.084438	6.197028	6.313752	9
81	6.313752	6.434843	6.560554	6.691156	6.826944	6.968234	7.115370	8
82	7.115370	7.268725	7.428706	7.595754	7.770351	7.953022	8.144346	7
83	8.144346	8.344956	8.555547	8.776887	9.009826	9.255304	9.514364	6
84	9.514364	9.788173	10.07803	10.38540	10.71191	11.05943	11.43005	5
85	11.43005	11.82617	12.25051	12.70620	13.19688	13.72674	14.30067	4
86	14.30067	14.92442	15.60478	16.34986	17.16934	18.07498	19.08114	3
87	19.08114	20.20555	21.47040	22.90377	24.54176	26.43160	28.63625	2
88	28.63625	31.24158	34.36777	38.18846	42.96408	49.10388	57.28996	1
89	57.28996	68.75009	85.93979	114.5887	171.8854	343.7737	∞	0
Deg.	60′	50′	40′	30′	20′	10′	0′	Deg.

NATURAL COTANGENTS.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	— ∞		— ∞		60	0	8.241855	119.6	8.241921	119.7	60
1	6.463726	5017	6.463726	5017	59	1	249033	117.7	249102	117.7	59
2	764756	2934	764756	2935	58	2	256094	115.8	256165	115.8	58
3	940847	2082	940847	2082	57	3	263042	114.0	263115	114.0	57
4	7.065786	1615	7.065786	1615	56	4	269881	112.2	269956	112.2	56
5	162696	1319	162696	1320	55	5	276614	110.5	276691	110.5	55
6	241877	1115	241878	1116	54	6	283243	108.8	283323	108.9	54
7	308824	966.5	308825	966.5	53	7	289773	107.2	289856	107.2	53
8	366816	852.5	366817	852.5	52	8	296207	105.6	296292	105.7	52
9	417968	762.6	417970	762.6	51	9	302546	104.1	302634	104.2	51
10	463726	689.8	463727	689.9	50	10	308794	102.7	308884	102.7	50
11	7.505118	629.8	7.505120	629.8	49	11	8.314954	101.2	8.315046	101.3	49
12	542906	579.3	542909	579.4	48	12	321027	99.82	321122	99.87	48
13	577668	536.4	577672	536.4	47	13	327016	98.47	327114	98.51	47
14	609853	499.3	609857	499.4	46	14	332924	97.14	333025	97.19	46
15	639816	467.1	639820	467.1	45	15	338753	95.86	338856	95.90	45
16	667845	438.8	667849	438.8	44	16	344504	94.60	344610	94.65	44
17	694173	413.7	694179	413.7	43	17	350181	93.38	350289	93.43	43
18	718997	391.3	719003	391.3	42	18	355783	92.19	355895	92.24	42
19	742478	371.2	742484	371.2	41	19	361315	91.03	361430	91.08	41
20	764754	353.1	764761	353.2	40	20	366777	89.90	366895	89.95	40
21	7.785943	336.7	7.785951	336.7	39	21	8.372171	88.80	8.372292	88.85	39
22	806146	321.7	806155	321.7	38	22	377499	87.72	377622	87.77	38
23	825451	308.0	825460	308.0	37	23	382762	86.67	382889	86.72	37
24	843934	295.4	843944	295.5	36	24	387962	85.64	388092	85.70	36
25	861662	283.9	861674	283.9	35	25	393101	84.64	393234	84.70	35
26	878695	273.2	878708	273.2	34	26	398179	83.66	398315	83.71	34
27	895085	263.2	895099	263.2	33	27	403199	82.71	403338	82.76	33
28	910879	254.0	910894	254.0	32	28	408161	81.77	408304	81.82	32
29	926119	245.3	926134	245.4	31	29	413068	80.86	413213	80.91	31
30	940842	237.3	940858	237.3	30	30	417919	79.96	418068	80.02	30
31	7.955082	229.8	7.955100	229.8	29	31	8.422717	79.09	8.422869	79.14	29
32	968870	222.7	968889	222.7	28	32	427462	78.23	427618	78.29	28
33	982233	216.1	982253	216.1	27	33	432156	77.40	432315	77.45	27
34	995198	209.8	995219	209.8	26	34	436800	76.57	436962	76.63	26
35	8.007787	203.9	8.007809	203.9	25	35	441394	75.77	441560	75.83	25
36	020021	198.3	020044	198.3	24	36	445941	74.99	446110	75.05	24
37	031919	193.0	031945	193.0	23	37	450440	74.22	450613	74.28	23
38	043501	188.0	043527	188.0	22	38	454893	73.46	455070	73.52	22
39	054781	183.2	054809	183.3	21	39	459301	72.73	459481	72.79	21
40	065776	178.7	065806	178.7	20	40	463665	72.00	463849	72.06	20
41	8.076500	174.4	8.076531	174.4	19	41	8.467985	71.29	8.468172	71.35	19
42	086935	170.3	086997	170.3	18	42	472263	70.60	472454	70.66	18
43	097183	166.4	097217	166.4	17	43	476498	69.91	476693	69.98	17
44	107167	162.6	107203	162.7	16	44	480693	69.24	480892	69.31	16
45	116926	159.1	116963	159.1	15	45	484848	68.59	485050	68.65	15
46	126471	155.6	126510	155.7	14	46	488963	67.94	489170	68.01	14
47	135810	152.4	135851	152.4	13	47	493040	67.31	493250	67.38	13
48	144953	149.2	144996	149.3	12	48	497078	66.69	497293	66.76	12
49	153907	146.2	153952	146.2	11	49	501080	66.08	501298	66.15	11
50	162681	143.3	162727	143.3	10	50	505045	65.48	505267	65.55	10
51	8.171280	140.5	8.171328	140.6	9	51	8.508974	64.89	8.509200	64.96	9
52	179713	137.8	179763	137.9	8	52	512867	64.31	513098	64.39	8
53	187985	135.3	188036	135.3	7	53	516726	63.75	516961	63.82	7
54	196102	132.8	196156	132.8	6	54	520551	63.19	520790	63.26	6
55	204070	130.4	204126	130.4	5	55	524343	62.64	524586	62.72	5
56	211895	128.1	211953	128.1	4	56	528102	62.11	528349	62.18	4
57	219581	125.9	219641	125.9	3	57	531828	61.58	532080	61.65	3
58	227134	123.7	227195	123.8	2	58	535523	61.06	535779	61.13	2
59	234557	121.6	234621	121.7	1	59	539186	60.55	539447	60.62	1
60	241855		241921		0	60	542819		543084		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1″	Tang.	PP1″	M.
0	8.542819	60.04	8.543084	60.12	60
1	546422	59.55	546691	59.62	59
2	549995	59.06	550268	59.14	58
3	553539	58.58	553817	58.66	57
4	557054	58.11	557336	58.19	56
5	560540	57.65	560828	57.73	55
6	563999	57.19	564291	57.27	54
7	567431	56.74	567727	56.82	53
8	570836	56.30	571137	56.38	52
9	574214	55.87	574520	55.95	51
10	577566	55.44	577877	55.52	50
11	8.580892	55.02	8.581208	55.10	49
12	584193	54.60	584514	54.68	48
13	587469	54.19	587795	54.27	47
14	590721	53.79	591051	53.87	46
15	593948	53.39	594283	53.47	45
16	597152	53.00	597492	53.08	44
17	600332	52.61	600677	52.70	43
18	603489	52.23	603839	52.32	42
19	606623	51.86	606978	51.94	41
20	609734	51.49	610094	51.58	40
21	8.612823	51.12	8.613189	51.21	39
22	615891	50.76	616262	50.85	38
23	618937	50.41	619313	50.50	37
24	621962	50.06	622343	50.15	36
25	624965	49.72	625352	49.81	35
26	627948	49.38	628340	49.47	34
27	630911	49.04	631308	49.13	33
28	633854	48.71	634256	48.80	32
29	636776	48.39	637184	48.48	31
30	639680	48.06	640093	48.16	30
31	8.642563	47.75	8.642982	47.84	29
32	645428	47.43	645853	47.53	28
33	648274	47.12	648704	47.22	27
34	651102	46.82	651537	46.91	26
35	653911	46.52	654352	46.61	25
36	656702	46.22	657149	46.31	24
37	659475	45.92	659928	46.02	23
38	662230	45.63	662689	45.73	22
39	664968	45.35	665433	45.44	21
40	667689	45.06	668160	45.16	20
41	8.670393	44.79	8.670870	44.88	19
42	673080	44.51	673563	44.61	18
43	675751	44.24	676239	44.34	17
44	678405	43.97	678900	44.07	16
45	681043	43.70	681544	43.80	15
46	683665	43.44	684172	43.54	14
47	686272	43.18	686784	43.28	13
48	688863	42.92	689381	43.03	12
49	691438	42.67	691963	42.77	11
50	693998	42.42	694529	42.52	10
51	8.696543	42.17	8.697081	42.28	9
52	699073	41.92	699617	42.03	8
53	701589	41.68	702139	41.79	7
54	704090	41.44	704646	41.55	6
55	706577	41.21	707140	41.32	5
56	709049	40.97	709618	41.08	4
57	711507	40.74	712083	40.85	3
58	713952	40.51	714534	40.62	2
59	716383	40.29	716972	40.40	1
60	718800		719396		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1″	Tang.	PP1″	M.
0	8.718800	40.06	8.719396	40.17	60
1	721204	39.84	721806	39.95	59
2	723595	39.62	724204	39.74	58
3	725972	39.41	726588	39.52	57
4	728337	39.19	728959	39.30	56
5	730688	38.98	731317	39.09	55
6	733027	38.77	733663	38.89	54
7	735354	38.57	735996	38.68	53
8	737667	38.36	738317	38.48	52
9	739969	38.16	740626	38.27	51
10	742259	37.96	742922	38.07	50
11	8.744536	37.76	8.745207	37.87	49
12	746802	37.56	747479	37.68	48
13	749055	37.37	749740	37.49	47
14	751297	37.17	751989	37.29	46
15	753528	36.98	754227	37.10	45
16	755747	36.80	756453	36.92	44
17	757955	36.61	758668	36.73	43
18	760151	36.42	760872	36.55	42
19	762337	36.24	763065	36.36	41
20	764511	36.06	765246	36.18	40
21	8.766675	35.88	8.767417	36.00	39
22	768828	35.70	769578	35.83	38
23	770970	35.53	771727	35.65	37
24	773101	35.35	773866	35.48	36
25	775223	35.18	775995	35.31	35
26	777333	35.01	778114	35.14	34
27	779434	34.84	780222	34.97	33
28	781524	34.67	782320	34.80	32
29	783605	34.51	784408	34.64	31
30	785675	34.35	786486	34.47	30
31	8.787736	34.18	8.788554	34.31	29
32	789787	34.02	790613	34.15	28
33	791828	33.86	792662	33.99	27
34	793859	33.70	794701	33.83	26
35	795881	33.54	796731	33.68	25
36	797894	33.39	798752	33.52	24
37	799897	33.23	800763	33.37	23
38	801892	33.08	802765	33.22	22
39	803876	32.93	804758	33.07	21
40	805852	32.78	806742	32.92	20
41	8.807819	32.63	8.808717	32.77	19
42	809777	32.49	810683	32.62	18
43	811726	32.34	812641	32.48	17
44	813667	32.19	814589	32.33	16
45	815599	32.05	816529	32.19	15
46	817522	31.91	818461	32.05	14
47	819436	31.77	820384	31.91	13
48	821343	31.63	822298	31.77	12
49	823240	31.49	824205	31.63	11
50	825130	31.35	826103	31.50	10
51	8.827011	31.22	8.827992	31.36	9
52	828884	31.08	829874	31.23	8
53	830749	30.95	831748	31.10	7
54	832607	30.82	833613	30.96	6
55	834456	30.69	835471	30.83	5
56	836297	30.56	837321	30.70	4
57	838130	30.43	839163	30.57	3
58	839956	30.30	840998	30.45	2
59	841774	30.17	842825	30.32	1
60	843585		844644		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1"	Tang.	PP1"	M.
0	8.843585	30.05	8.844644	30.19	60
1	845387	29.92	846455	30.07	59
2	847183	29.80	848260	29.95	58
3	848971	29.67	850057	29.82	57
4	850751	29.55	851846	29.70	56
5	852525	29.43	853628	29.58	55
6	854291	29.31	855403	29.46	54
7	856049	29.19	857171	29.35	53
8	857801	29.07	858932	29.23	52
9	859546	28.96	860686	29.11	51
10	861283	28.84	862433	29.00	50
11	8.863014	28.73	8.864173	28.88	49
12	864738	28.61	865906	28.77	48
13	866455	28.50	867632	28.66	47
14	868165	28.39	869351	28.54	46
15	869868	28.28	871064	28.43	45
16	871565	28.17	872770	28.32	44
17	873255	28.06	874469	28.21	43
18	874938	27.95	876162	28.11	42
19	876615	27.84	877849	28.00	41
20	878285	27.73	879529	27.89	40
21	8.879949	27.63	8.881202	27.79	39
22	881607	27.52	882869	27.68	38
23	883258	27.42	884530	27.58	37
24	884903	27.31	886185	27.47	36
25	886542	27.21	887833	27.37	35
26	888174	27.11	889476	27.27	34
27	889801	27.00	891112	27.17	33
28	891421	26.90	892742	27.07	32
29	893035	26.80	894366	26.97	31
30	894643	26.70	895984	26.87	30
31	8.896246	26.60	8.897596	26.77	29
32	897842	26.51	899203	26.67	28
33	899432	26.41	900803	26.58	27
34	901017	26.31	902398	26.48	26
35	902596	26.22	903987	26.38	25
36	904169	26.12	905570	26.29	24
37	905736	26.03	907147	26.20	23
38	907297	25.93	908719	26.10	22
39	908853	25.84	910285	26.01	21
40	910404	25.75	911846	25.92	20
41	8.911949	25.66	8.913401	25.83	19
42	913488	25.56	914951	25.74	18
43	915022	25.47	916495	25.65	17
44	916550	25.38	918034	25.56	16
45	918073	25.29	919568	25.47	15
46	919591	25.20	921096	25.38	14
47	921103	25.12	922619	25.30	13
48	922610	25.03	924136	25.21	12
49	924112	24.91	925649	25.12	11
50	925609	24.86	927156	25.03	10
51	8.927100	24.77	8.928658	24.95	9
52	928587	24.69	930155	24.87	8
53	930068	24.60	931647	24.78	7
54	931544	24.52	933134	24.70	6
55	933015	24.43	934616	24.62	5
56	934481	24.35	936093	24.53	4
57	935942	24.27	937565	24.45	3
58	937398	24.19	939032	24.37	2
59	938850	24.11	940494	24.29	1
60	940296		941952		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1"	Tang.	PP1"	M.
0	8.940296	24.03	8.941952	24.21	60
1	941738	23.94	943404	24.13	59
2	943174	23.87	944852	24.05	58
3	944606	23.79	946295	23.97	57
4	946034	23.71	947734	23.90	56
5	947456	23.63	949168	23.82	55
6	948874	23.55	950597	23.74	54
7	950287	23.48	952021	23.66	53
8	951696	23.40	953441	23.60	52
9	953100	23.32	954856	23.51	51
10	954499	23.25	956267	23.44	50
11	8.955894	23.17	8.957674	23.37	49
12	957284	23.10	959075	23.29	48
13	958670	23.02	960473	23.22	47
14	960052	22.95	961866	23.14	46
15	961429	22.88	963255	23.07	45
16	962801	22.80	964639	23.00	44
17	964170	22.73	966019	22.93	43
18	965534	22.66	967394	22.86	42
19	966893	22.59	968766	22.79	41
20	968249	22.52	970133	22.71	40
21	8.969600	22.45	8.971496	22.65	39
22	970947	22.38	972855	22.57	38
23	972289	22.31	974209	22.51	37
24	973628	22.24	975560	22.44	36
25	974962	22.17	976906	22.37	35
26	976293	22.10	978248	22.30	34
27	977619	22.03	979586	22.23	33
28	978941	21.97	980921	22.17	32
29	980259	21.90	982251	22.10	31
30	981573	21.83	983577	22.04	30
31	8.982883	21.77	8.984899	21.97	29
32	984189	21.70	986217	21.91	28
33	985491	21.63	987532	21.84	27
34	986789	21.57	988842	21.78	26
35	988083	21.50	990149	21.71	25
36	989374	21.44	991451	21.65	24
37	990660	21.38	992750	21.58	23
38	991943	21.31	994045	21.52	22
39	993222	21.25	995337	21.46	21
40	994497	21.19	996624	21.40	20
41	8.995768	21.12	8.997908	21.34	19
42	997036	21.06	999188	21.27	18
43	998299	21.00	9.000465	21.21	17
44	999560	20.94	001738	21.15	16
45	9.000816	20.88	003007	21.09	15
46	002069	20.82	004272	21.03	14
47	003318	20.76	005534	20.97	13
48	004563	20.70	006792	20.91	12
49	005805	20.64	008047	20.85	11
50	007044	20.58	009298	20.80	10
51	9.008278	20.52	9.010546	20.74	9
52	009510	20.46	011790	20.68	8
53	010737	20.40	013031	20.62	7
54	011962	20.34	014268	20.56	6
55	013182	20.29	015502	20.51	5
56	014400	20.23	016732	20.45	4
57	015613	20.17	017959	20.40	3
58	016824	20.12	019183	20.33	2
59	018031	20.06	020403	20.28	1
60	019235		021620		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.019235	20.00	9.021620	20.23	60	0	9.085894	17.13	9.089144	17.38	60
1	020435	19.95	022834	20.17	59	1	086922	17.09	090187	17.34	59
2	021632	19.89	024044	20.11	58	2	087947	17.04	091228	17.30	58
3	022825	19.84	025251	20.06	57	3	088970	17.00	092266	17.27	57
4	024016	19.78	026455	20.00	56	4	089990	16.96	093302	17.22	56
5	025203	19.73	027655	19.95	55	5	091008	16.92	094336	17.19	55
6	026386	19.67	028852	19.90	54	6	092024	16.88	095367	17.15	54
7	027567	19.62	030046	19.85	53	7	093037	16.84	096395	17.11	53
8	028744	19.57	031237	19.79	52	8	094047	16.80	097422	17.07	52
9	029918	19.51	032425	19.74	51	9	095056	16.76	098446	17.03	51
10	031089	19.46	033609	19.69	50	10	096062	16.73	099468	16.99	50
11	9.032257	19.41	9.034791	19.64	49	11	9.097065	16.68	9.100487	16.95	49
12	033421	19.35	035969	19.58	48	12	098066	16.65	101504	16.91	48
13	034582	19.30	037144	19.53	47	13	099065	16.61	102519	16.87	47
14	035741	19.25	038316	19.48	46	14	100062	16.57	103532	16.84	46
15	036896	19.20	039485	19.43	45	15	101056	16.53	104542	16.80	45
16	038048	19.15	040651	19.38	44	16	102048	16.49	105550	16.76	44
17	039197	19.10	041813	19.33	43	17	103037	16.45	106556	16.72	43
18	040342	19.05	042973	19.28	42	18	104025	16.41	107559	16.69	42
19	041485	18.99	044130	19.23	41	19	105010	16.38	108560	16.65	41
20	042625	18.94	045284	19.18	40	20	105992	16.34	109559	16.61	40
21	9.043762	18.89	9.046434	19.13	39	21	9.106973	16.30	9.110556	16.58	39
22	044895	18.84	047582	19.08	38	22	107951	16.27	111551	16.54	38
23	046026	18.80	048727	19.03	37	23	108927	16.23	112543	16.50	37
24	047154	18.75	049869	18.98	36	24	109901	16.19	113533	16.46	36
25	048279	18.70	051008	18.93	35	25	110873	16.16	114521	16.43	35
26	049400	18.65	052144	18.89	34	26	111842	16.12	115507	16.39	34
27	050519	18.60	053277	18.84	33	27	112809	16.08	116491	16.36	33
28	051635	18.55	054407	18.79	32	28	113774	16.05	117472	16.32	32
29	052749	18.50	055535	18.74	31	29	114737	16.01	118452	16.29	31
30	053859	18.45	056659	18.70	30	30	115698	15.97	119429	16.25	30
31	9.054966	18.41	9.057781	18.65	29	31	9.116656	15.94	9.120404	16.22	29
32	056071	18.36	058900	18.60	28	32	117613	15.90	121377	16.18	28
33	057172	18.31	060016	18.55	27	33	118567	15.87	122348	16.15	27
34	058271	18.27	061130	18.51	26	34	119519	15.83	123317	16.11	26
35	059367	18.22	062240	18.46	25	35	120469	15.80	124284	16.07	25
36	060460	18.17	063348	18.42	24	36	121417	15.76	125249	16.04	24
37	061551	18.13	064453	18.37	23	37	122362	15.73	126211	16.01	23
38	062639	18.08	065556	18.33	22	38	123306	15.69	127172	15.97	22
39	063724	18.04	066655	18.28	21	39	124248	15.66	128130	15.94	21
40	064806	17.99	067752	18.24	20	40	125187	15.62	129087	15.91	20
41	9.065885	17.94	9.068846	18.19	19	41	9.126125	15.59	9.130041	15.87	19
42	066962	17.90	069938	18.15	18	42	127060	15.56	130994	15.84	18
43	068036	17.86	071027	18.10	17	43	127993	15.52	131944	15.81	17
44	069107	17.81	072113	18.06	16	44	128925	15.49	132893	15.77	16
45	070176	17.77	073197	18.02	15	45	129854	15.45	133839	15.74	15
46	071242	17.72	074278	17.97	14	46	130781	15.42	134784	15.71	14
47	072306	17.68	075356	17.93	13	47	131706	15.39	135726	15.67	13
48	073366	17.63	076432	17.89	12	48	132630	15.35	136667	15.64	12
49	074424	17.59	077505	17.84	11	49	133551	15.32	137605	15.61	11
50	075480	17.55	078576	17.80	10	50	134470	15.29	138542	15.58	10
51	9.076533	17.50	9.079644	17.76	9	51	9.135387	15.25	9.139476	15.55	9
52	077583	17.46	080710	17.72	8	52	136303	15.22	140409	15.51	8
53	078631	17.42	081773	17.67	7	53	137216	15.19	141340	15.48	7
54	079676	17.38	082833	17.63	6	54	138128	15.16	142269	15.45	6
55	080719	17.33	083891	17.59	5	55	139037	15.12	143196	15.42	5
56	081759	17.29	084947	17.55	4	56	139944	15.09	144121	15.39	4
57	082797	17.25	086000	17.51	3	57	140850	15.06	145044	15.35	3
58	083832	17.21	087050	17.47	2	58	141754	15.03	145966	15.32	2
59	084864	17.17	088098	17.43	1	59	142655	15.00	146885	15.29	1
60	085894		089144		0	60	143555		147803		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.143555	14.96	9.147803	15.26	60
1	144453	14.93	148718	15.23	59
2	145349	14.90	149632	15.20	58
3	146243	14.87	150544	15.17	57
4	147136	14.84	151454	15.14	56
5	148026	14.81	152363	15.11	55
6	148915	14.78	153269	15.08	54
7	149802	14.75	154174	15.05	53
8	150686	14.72	155077	15.02	52
9	151569	14.69	155978	14.99	51
10	152451	14.66	156877	14.96	50
11	9.153330	14.63	9.157775	14.93	49
12	154208	14.60	158671	14.90	48
13	155083	14.57	159565	14.87	47
14	155957	14.54	160457	14.84	46
15	156830	14.51	161347	14.81	45
16	157700	14.48	162236	14.79	44
17	158569	14.45	163123	14.76	43
18	159435	14.42	164008	14.73	42
19	160301	14.39	164892	14.70	41
20	161164	14.36	165774	14.67	40
21	9.162025	14.33	9.166654	14.64	39
22	162885	14.30	167532	14.61	38
23	163743	14.27	168409	14.58	37
24	164600	14.24	169284	14.55	36
25	165454	14.22	170157	14.53	35
26	166307	14.19	171029	14.50	34
27	167159	14.16	171899	14.47	33
28	168008	14.13	172767	14.44	32
29	168856	14.10	173634	14.42	31
30	169702	14.07	174499	14.39	30
31	9.170547	14.05	9.175362	14.36	29
32	171389	14.02	176224	14.33	28
33	172230	13.99	177084	14.31	27
34	173070	13.96	177942	14.28	26
35	173908	13.91	178799	14.25	25
36	174744	13.91	179655	14.22	24
37	175578	13.88	180508	14.20	23
38	176411	13.86	181360	14.17	22
39	177242	13.83	182211	14.15	21
40	178072	13.80	183059	14.12	20
41	9.178900	13.77	9.183907	14.09	19
42	179726	13.74	184752	14.07	18
43	180551	13.72	185597	14.04	17
44	181374	13.69	186439	14.02	16
45	182196	13.66	187280	13.99	15
46	183016	13.64	188120	13.96	14
47	183834	13.61	188958	13.93	13
48	184651	13.59	189794	13.91	12
49	185466	13.56	190629	13.89	11
50	186280	13.53	191462	13.86	10
51	9.187092	13.51	9.192294	13.84	9
52	187903	13.48	193124	13.81	8
53	188712	13.46	193953	13.79	7
54	189519	13.43	194780	13.76	6
55	190325	13.41	195606	13.74	5
56	191130	13.38	196430	13.71	4
57	191933	13.36	197253	13.69	3
58	192734	13.33	198074	13.66	2
59	193534	13.30	198894	13.64	1
60	194332		199713		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.194332	13.28	9.199713	13.61	60
1	195129	13.26	200529	13.59	59
2	195925	13.23	201345	13.57	58
3	196719	13.21	202159	13.54	57
4	197511	13.18	202971	13.52	56
5	198302	13.16	203782	13.49	55
6	199091	13.13	204592	13.47	54
7	199879	13.11	205400	13.45	53
8	200666	13.08	206207	13.42	52
9	201451	13.06	207013	13.40	51
10	202234	13.04	207817	13.38	50
11	9.203017	13.01	9.208619	13.35	49
12	203797	12.99	209420	13.33	48
13	204577	12.96	210220	13.31	47
14	205354	12.94	211018	13.28	46
15	206131	12.92	211815	13.26	45
16	206906	12.89	212611	13.24	44
17	207679	12.87	213405	13.21	43
18	208452	12.85	214198	13.19	42
19	209222	12.83	214989	13.17	41
20	209992	12.80	215780	13.15	40
21	9.210760	12.78	9.216568	13.12	39
22	211526	12.75	217356	13.10	38
23	212291	12.73	218142	13.08	37
24	213055	12.71	218926	13.06	36
25	213818	12.68	219710	13.03	35
26	214579	12.66	220492	13.01	34
27	215338	12.64	221272	12.99	33
28	216097	12.62	222052	12.97	32
29	216854	12.59	222830	12.94	31
30	217609	12.57	223607	12.92	30
31	9.218363	12.55	9.224382	12.90	29
32	219116	12.53	225156	12.88	28
33	219868	12.50	225929	12.86	27
34	220618	12.48	226700	12.84	26
35	221367	12.46	227471	12.81	25
36	222115	12.44	228239	12.79	24
37	222861	12.42	229007	12.77	23
38	223606	12.39	229773	12.75	22
39	224349	12.37	230539	12.73	21
40	225092	12.35	231302	12.71	20
41	9.225833	12.33	9.232065	12.69	19
42	226573	12.31	232826	12.67	18
43	227311	12.28	233586	12.65	17
44	228048	12.26	234345	12.62	16
45	228784	12.24	235103	12.60	15
46	229518	12.22	235859	12.58	14
47	230252	12.20	236614	12.56	13
48	230984	12.18	237368	12.54	12
49	231714	12.16	238120	12.52	11
50	232444	12.14	238872	12.50	10
51	9.233172	12.12	9.239622	12.48	9
52	233899	12.09	240371	12.46	8
53	234625	12.07	241118	12.44	7
54	235349	12.05	241865	12.42	6
55	236073	12.03	242610	12.40	5
56	236795	12.01	243354	12.38	4
57	237515	11.99	244097	12.36	3
58	238235	11.97	244839	12.34	2
59	238953	11.95	245579	12.32	1
60	239670		246319		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.239670	11.93	9.246319	12.30	60	0	9.280599	10.82	9.288652	11.23	60
1	240386	11.91	247057	12.28	59	1	281248	10.81	289326	11.22	59
2	241101	11.89	247794	12.26	58	2	281897	10.79	289999	11.20	58
3	241814	11.87	248530	12.24	57	3	282544	10.77	290671	11.18	57
4	242526	11.85	249264	12.22	56	4	283190	10.76	291342	11.17	56
5	243237	11.83	249998	12.20	55	5	283836	10.74	292013	11.15	55
6	243947	11.81	250730	12.18	54	6	284480	10.72	292682	11.14	54
7	244656	11.79	251461	12.17	53	7	285124	10.71	293350	11.12	53
8	245363	11.77	252191	12.15	52	8	285766	10.69	294017	11.11	52
9	246069	11.75	252920	12.13	51	9	286408	10.67	294684	11.09	51
10	246775	11.73	253648	12.11	50	10	287048	10.66	295349	11.07	50
11	9.247478	11.71	9.254374	12.09	49	11	9.287688	10.64	9.296013	11.06	49
12	248181	11.69	255100	12.07	48	12	288326	10.63	296677	11.04	48
13	248883	11.67	255824	12.05	47	13	288964	10.61	297339	11.03	47
14	249583	11.65	256547	12.03	46	14	289600	10.59	298001	11.01	46
15	250282	11.63	257269	12.01	45	15	290236	10.58	298662	11.00	45
16	250980	11.61	257990	12.00	44	16	290870	10.56	299322	10.98	44
17	251677	11.59	258710	11.98	43	17	291504	10.54	299980	10.96	43
18	252373	11.58	259429	11.96	42	18	292137	10.53	300638	10.95	42
19	253067	11.56	260146	11.94	41	19	292768	10.51	301295	10.93	41
20	253761	11.54	260863	11.92	40	20	293399	10.50	301951	10.92	40
21	9.254453	11.52	9.261578	11.90	39	21	9.294029	10.48	9.302607	10.90	39
22	255144	11.50	262292	11.89	38	22	294658	10.46	303261	10.89	38
23	255834	11.48	263005	11.87	37	23	295286	10.45	303914	10.87	37
24	256523	11.46	263717	11.85	36	24	295913	10.43	304567	10.86	36
25	257211	11.44	264428	11.83	35	25	296539	10.42	305218	10.84	35
26	257898	11.42	265138	11.81	34	26	297164	10.40	305869	10.83	34
27	258583	11.41	265847	11.79	33	27	297788	10.39	306519	10.81	33
28	259268	11.39	266555	11.78	32	28	298412	10.37	307168	10.80	32
29	259951	11.37	267261	11.76	31	29	299034	10.36	307816	10.78	31
30	260633	11.35	267967	11.74	30	30	299655	10.34	308463	10.77	30
31	9.261314	11.33	9.268671	11.72	29	31	9.300276	10.32	9.309109	10.75	29
32	261994	11.31	269375	11.70	28	32	300895	10.31	309754	10.74	28
33	262673	11.30	270077	11.69	27	33	301514	10.29	310399	10.73	27
34	263351	11.28	270779	11.67	26	34	302132	10.28	311042	10.71	26
35	264027	11.26	271479	11.65	25	35	302748	10.26	311685	10.70	25
36	264703	11.24	272178	11.64	24	36	303364	10.25	312327	10.68	24
37	265377	11.22	272876	11.62	23	37	303979	10.23	312968	10.67	23
38	266051	11.20	273573	11.60	22	38	304593	10.22	313608	10.65	22
39	266723	11.19	274269	11.58	21	39	305207	10.20	314247	10.64	21
40	267395	11.17	274964	11.57	20	40	305819	10.19	314885	10.62	20
41	9.268065	11.15	9.275658	11.55	19	41	9.306430	10.17	9.315523	10.61	19
42	268734	11.13	276351	11.53	18	42	307041	10.16	316159	10.60	18
43	269402	11.12	277043	11.51	17	43	307650	10.14	316795	10.58	17
44	270069	11.10	277734	11.50	16	44	308259	10.13	317430	10.57	16
45	270735	11.08	278424	11.48	15	45	308867	10.11	318064	10.55	15
46	271400	11.06	279113	11.47	14	46	309474	10.10	318697	10.54	14
47	272064	11.05	279801	11.45	13	47	310080	10.08	319330	10.53	13
48	272726	11.03	280488	11.43	12	48	310685	10.07	319961	10.51	12
49	273388	11.01	281174	11.41	11	49	311289	10.06	320592	10.50	11
50	274049	10.99	281858	11.40	10	50	311893	10.04	321222	10.48	10
51	9.274708	10.98	9.282542	11.38	9	51	9.312495	10.03	9.321851	10.47	9
52	275367	10.96	283225	11.36	8	52	313097	10.01	322479	10.45	8
53	276025	10.94	283907	11.35	7	53	313698	10.00	323106	10.44	7
54	276681	10.92	284588	11.33	6	54	314297	9.98	323733	10.43	6
55	277337	10.91	285268	11.31	5	55	314897	9.97	324358	10.41	5
56	277991	10.89	285947	11.30	4	56	315495	9.96	324983	10.40	4
57	278645	10.87	286624	11.28	3	57	316092	9.94	325607	10.39	3
58	279297	10.86	287301	11.26	2	58	316689	9.93	326231	10.37	2
59	279948	10.84	287977	11.25	1	59	317284	9.91	326853	10.36	1
60	280599		288652		0	60	317879		327475		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.317879	9.90	9.327475	10.35	60
1	318473	9.88	328095	10.33	59
2	319066	9.87	328715	10.32	58
3	319658	9.86	329334	10.30	57
4	320249	9.84	329953	10.29	56
5	320840	9.83	330570	10.28	55
6	321430	9.82	331187	10.26	54
7	322019	9.80	331803	10.25	53
8	322607	9.79	332418	10.24	52
9	323194	9.77	333033	10.22	51
10	323780	9.76	333646	10.21	50
11	9.324366	9.75	9.334259	10.20	49
12	324950	9.73	334871	10.19	48
13	325534	9.72	335482	10.17	47
14	326117	9.70	336093	10.16	46
15	326700	9.69	336702	10.15	45
16	327281	9.68	337311	10.13	44
17	327862	9.66	337919	10.12	43
18	328442	9.65	338527	10.11	42
19	329021	9.64	339133	10.10	41
20	329599	9.62	339739	10.08	40
21	9.330176	9.61	9.340344	10.07	39
22	330753	9.60	340948	10.06	38
23	331329	9.58	341552	10.04	37
24	331903	9.57	342155	10.03	36
25	332478	9.56	342757	10.02	35
26	333051	9.54	343358	10.00	34
27	333624	9.53	343958	9.99	33
28	334195	9.52	344558	9.98	32
29	334767	9.50	345157	9.97	31
30	335337	9.49	345755	9.96	30
31	9.335906	9.48	9.346353	9.94	29
32	336475	9.46	346949	9.93	28
33	337043	9.45	347545	9.92	27
34	337610	9.44	348141	9.91	26
35	338176	9.43	348735	9.90	25
36	338742	9.41	349329	9.88	24
37	339307	9.40	349922	9.87	23
38	339871	9.39	350514	9.86	22
39	340434	9.37	351106	9.85	21
40	340996	9.36	351697	9.83	20
41	9.341558	9.35	9.352287	9.82	19
42	342119	9.34	352876	9.81	18
43	342679	9.32	353465	9.80	17
44	343239	9.31	354053	9.79	16
45	343797	9.30	354640	9.77	15
46	344355	9.29	355227	9.76	14
47	344912	9.27	355813	9.75	13
48	345469	9.26	356398	9.74	12
49	346024	9.25	356982	9.73	11
50	346579	9.24	357566	9.71	10
51	9.347134	9.22	9.358149	9.70	9
52	347687	9.21	358731	9.69	8
53	348240	9.20	359313	9.68	7
54	348792	9.19	359893	9.67	6
55	349343	9.17	360474	9.66	5
56	349893	9.16	361053	9.65	4
57	350443	9.15	361632	9.63	3
58	350992	9.14	362210	9.62	2
59	351540	9.13	362787	9.61	1
60	352088		363364		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.352088	9.11	9.363364	9.60	60
1	352635	9.10	363940	9.59	59
2	353181	9.09	364515	9.58	58
3	353726	9.08	365090	9.57	57
4	354271	9.07	365664	9.55	56
5	354815	9.05	366237	9.54	55
6	355358	9.04	366810	9.53	54
7	355901	9.03	367382	9.52	53
8	356443	9.02	367953	9.51	52
9	356984	9.01	368524	9.50	51
10	357524	8.99	369094	9.49	50
11	9.358064	8.98	9.369663	9.48	49
12	358603	8.97	370232	9.46	48
13	359141	8.96	370799	9.45	47
14	359678	8.95	371367	9.44	46
15	360215	8.93	371933	9.43	45
16	360752	8.92	372499	9.42	44
17	361287	8.91	373064	9.41	43
18	361822	8.90	373629	9.40	42
19	362356	8.89	374193	9.39	41
20	362889	8.88	374756	9.38	40
21	9.363422	8.87	9.375319	9.37	39
22	363954	8.85	375881	9.35	38
23	364485	8.84	376442	9.34	37
24	365016	8.83	377003	9.33	36
25	365546	8.82	377563	9.32	35
26	366075	8.81	378122	9.31	34
27	366604	8.80	378681	9.30	33
28	367131	8.79	379239	9.29	32
29	367659	8.77	379797	9.28	31
30	368185	8.76	380354	9.27	30
31	9.368711	8.75	9.380910	9.26	29
32	369236	8.74	381466	9.25	28
33	369761	8.73	382020	9.24	27
34	370285	8.72	382575	9.23	26
35	370808	8.71	383129	9.22	25
36	371330	8.70	383682	9.21	24
37	371852	8.69	384234	9.20	23
38	372373	8.67	384786	9.19	22
39	372894	8.66	385337	9.18	21
40	373414	8.65	385888	9.17	20
41	9.373933	8.64	9.386438	9.15	19
42	374452	8.63	386987	9.14	18
43	374970	8.62	387536	9.13	17
44	375487	8.61	388084	9.12	16
45	376003	8.60	388631	9.11	15
46	376519	8.59	389178	9.10	14
47	377035	8.58	389724	9.09	13
48	377549	8.57	390270	9.08	12
49	378063	8.56	390815	9.07	11
50	378577	8.54	391360	9.06	10
51	9.379089	8.53	9.391903	9.05	9
52	379601	8.52	392447	9.04	8
53	380113	8.51	392989	9.03	7
54	380624	8.50	393531	9.02	6
55	381134	8.49	394073	9.01	5
56	381643	8.48	394614	9.00	4
57	382152	8.47	395154	8.99	3
58	382661	8.46	395694	8.98	2
59	383168	8.45	396233	8.97	1
60	383675		396771		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.383675	8.44	9.396771	8.96	60	0	9.412996	7.85	9.428052	8.42	60
1	384182	8.43	397309	8.96	59	1	413467	7.84	428558	8.41	59
2	384687	8.42	397846	8.95	58	2	413938	7.83	429062	8.40	58
3	385192	8.41	398383	8.94	57	3	414408	7.83	429566	8.39	57
4	385697	8.40	398919	8.93	56	4	414878	7.82	430070	8.38	56
5	386201	8.39	399455	8.92	55	5	415347	7.81	430573	8.38	55
6	386704	8.38	399990	8.91	54	6	415815	7.80	431075	8.37	54
7	387207	8.37	400524	8.90	53	7	416283	7.79	431577	8.36	53
8	387709	8.36	401058	8.89	52	8	416751	7.78	432079	8.35	52
9	388210	8.35	401591	8.88	51	9	417217	7.77	432580	8.34	51
10	388711	8.34	402124	8.87	50	10	417684	7.76	433080	8.33	50
11	9.389211	8.33	9.402656	8.86	49	11	9.418150	7.75	9.433580	8.32	49
12	389711	8.32	403187	8.85	48	12	418615	7.74	434080	8.32	48
13	390210	8.31	403718	8.84	47	13	419079	7.73	434579	8.31	47
14	390708	8.30	404249	8.83	46	14	419544	7.73	435078	8.30	46
15	391206	8.28	404778	8.82	45	15	420007	7.72	435576	8.29	45
16	391703	8.27	405308	8.81	44	16	420470	7.71	436073	8.28	44
17	392199	8.26	405836	8.80	43	17	420933	7.70	436570	8.28	43
18	392695	8.25	406364	8.79	42	18	421395	7.69	437067	8.27	42
19	393191	8.24	406892	8.78	41	19	421857	7.68	437563	8.26	41
20	393685	8.23	407419	8.77	40	20	422318	7.67	438059	8.25	40
21	9.394179	8.22	9.407945	8.76	39	21	9.422778	7.67	9.438554	8.24	39
22	394673	8.21	408471	8.75	38	22	423238	7.66	439048	8.23	38
23	395166	8.20	408996	8.74	37	23	423697	7.65	439543	8.23	37
24	395658	8.20	409521	8.74	36	24	424156	7.64	440036	8.22	36
25	396150	8.18	410045	8.73	35	25	424615	7.63	440529	8.21	35
26	396641	8.17	410569	8.72	34	26	425073	7.62	441022	8.20	34
27	397132	8.17	411092	8.71	33	27	425530	7.61	441514	8.19	33
28	397621	8.16	411615	8.70	32	28	425987	7.60	442006	8.19	32
29	398111	8.15	412137	8.69	31	29	426443	7.60	442497	8.18	31
30	398600	8.14	412658	8.68	30	30	426899	7.59	442988	8.17	30
31	9.399088	8.13	9.413179	8.67	29	31	9.427354	7.58	9.443479	8.16	29
32	399575	8.12	413699	8.66	28	32	427809	7.57	443968	8.16	28
33	400062	8.11	414219	8.65	27	33	428263	7.56	444458	8.15	27
34	400549	8.10	414738	8.64	26	34	428717	7.55	444947	8.14	26
35	401035	8.09	415257	8.64	25	35	429170	7.54	445435	8.13	25
36	401520	8.08	415775	8.63	24	36	429623	7.53	445923	8.12	24
37	402005	8.07	416293	8.62	23	37	430075	7.53	446411	8.12	23
38	402489	8.03	416810	8.61	22	38	430527	7.52	446898	8.11	22
39	402972	8.05	417326	8.60	21	39	430978	7.51	447384	8.10	21
40	403455	8.04	417842	8.59	20	40	431429	7.50	447870	8.09	20
41	9.403938	8.03	9.418358	8.58	19	41	9.431879	7.50	9.448356	8.09	19
42	404420	8.02	418873	8.57	18	42	432329	7.49	448841	8.08	18
43	404901	8.01	419387	8.56	17	43	432778	7.48	449326	8.07	17
44	405382	8.00	419901	8.55	16	44	433226	7.47	449810	8.06	16
45	405862	7.99	420415	8.55	15	45	433675	7.46	450294	8.06	15
46	406341	7.98	420927	8.54	14	46	434122	7.45	450777	8.05	14
47	406820	7.97	421440	8.53	13	47	434569	7.45	451260	8.04	13
48	407299	7.96	421952	8.52	12	48	435016	7.44	451743	8.03	12
49	407777	7.95	422463	8.51	11	49	435462	7.43	452225	8.02	11
50	408254	7.94	422974	8.50	10	50	435908	7.42	452706	8.02	10
51	9.408731	7.94	9.423481	8.49	9	51	9.436353	7.41	9.453187	8.01	9
52	409207	7.93	423993	8.48	8	52	436798	7.40	453668	8.00	8
53	409682	7.92	424503	8.48	7	53	437242	7.40	454148	7.99	7
54	410157	7.91	425011	8.47	6	54	437686	7.39	454628	7.99	6
55	410632	7.90	425519	8.46	5	55	438129	7.38	455107	7.98	5
56	411106	7.89	426027	8.45	4	56	438572	7.37	455586	7.97	4
57	411579	7.88	426534	8.44	3	57	439014	7.36	456064	7.96	3
58	412052	7.87	427041	8.43	2	58	439456	7.35	456542	7.96	2
59	412524	7.86	427547	8.43	1	59	439897	7.35	457019	7.95	1
60	412996		428052		0	60	440338		457496		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.440338	7.34	9.457496	7.94	60
1	440778	7.33	457973	7.93	59
2	441218	7.32	458449	7.93	58
3	441658	7.31	458925	7.92	57
4	442096	7.31	459400	7.91	56
5	442535	7.30	459875	7.90	55
6	442973	7.29	460349	7.90	54
7	443410	7.28	460823	7.89	53
8	443847	7.27	461297	7.88	52
9	444284	7.27	461770	7.88	51
10	444720	7.26	462242	7.87	50
11	9.445155	7.25	9.462715	7.86	49
12	445590	7.24	463186	7.85	48
13	446025	7.23	463658	7.85	47
14	446459	7.23	464128	7.84	46
15	446893	7.22	464599	7.83	45
16	447326	7.21	465069	7.83	44
17	447759	7.20	465539	7.82	43
18	448191	7.20	466008	7.81	42
19	448623	7.19	466477	7.80	41
20	449054	7.18	466945	7.80	40
21	9.449485	7.17	9.467413	7.79	39
22	449915	7.17	467880	7.78	38
23	450345	7.16	468347	7.78	37
24	450775	7.15	468814	7.77	36
25	451204	7.14	469280	7.76	35
26	451632	7.13	469746	7.75	34
27	452060	7.13	470211	7.75	33
28	452488	7.12	470676	7.74	32
29	452915	7.11	471141	7.73	31
30	453342	7.10	471605	7.73	30
31	9.453768	7.10	9.472069	7.72	29
32	454194	7.09	472532	7.71	28
33	454619	7.08	472995	7.70	27
34	455044	7.07	473457	7.70	26
35	455469	7.07	473919	7.70	25
36	455893	7.06	474381	7.69	24
37	456316	7.05	474842	7.68	23
38	456739	7.05	475303	7.67	22
39	457162	7.04	475763	7.67	21
40	457584	7.03	476223	7.66	20
41	9.458006	7.02	9.476683	7.65	19
42	458427	7.01	477142	7.65	18
43	458848	7.00	477601	7.64	17
44	459268	7.00	478059	7.63	16
45	459688	6.99	478517	7.63	15
46	460108	6.98	478975	7.62	14
47	460527	6.98	479432	7.61	13
48	460946	6.97	479889	7.60	12
49	461364	6.96	480345	7.60	11
50	461782	6.95	480801	7.60	10
51	9.462199	6.95	9.481257	7.59	9
52	462616	6.94	481712	7.58	8
53	463032	6.93	482167	7.57	7
54	463448	6.93	482621	7.57	6
55	463864	6.92	483075	7.56	5
56	464279	6.91	483529	7.55	4
57	464694	6.90	483982	7.55	3
58	465108	6.90	484435	7.54	2
59	465522	6.89	484887	7.53	1
60	465935		485339		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.465935	6.88	9.485339	7.53	60
1	466348	6.88	485791	7.52	59
2	466761	6.87	486242	7.51	58
3	467173	6.86	486693	7.51	57
4	467585	6.85	487143	7.50	56
5	467996	6.85	487593	7.49	55
6	468407	6.84	488043	7.49	54
7	468817	6.83	488492	7.48	53
8	469227	6.83	488941	7.47	52
9	469637	6.82	489390	7.47	51
10	470046	6.81	489838	7.46	50
11	9.470455	6.80	9.490286	7.46	49
12	470863	6.80	490733	7.45	48
13	471271	6.79	491180	7.44	47
14	471679	6.78	491627	7.44	46
15	472086	6.78	492073	7.43	45
16	472492	6.77	492519	7.43	44
17	472898	6.77	492965	7.42	43
18	473304	6.76	493410	7.41	42
19	473710	6.75	493854	7.40	41
20	474115	6.74	494299	7.40	40
21	9.474519	6.74	9.494743	7.40	39
22	474923	6.73	495186	7.39	38
23	475327	6.72	495630	7.38	37
24	475730	6.72	496073	7.37	36
25	476133	6.71	496515	7.37	35
26	476536	6.70	496957	7.36	34
27	476938	6.70	497399	7.36	33
28	477340	6.69	497841	7.35	32
29	477741	6.68	498282	7.34	31
30	478142	6.67	498722	7.34	30
31	9.478542	6.67	9.499163	7.33	29
32	478942	6.66	499603	7.33	28
33	479342	6.65	500042	7.32	27
34	479741	6.65	500481	7.31	26
35	480140	6.64	500920	7.31	25
36	480539	6.63	501359	7.30	24
37	480937	6.63	501797	7.30	23
38	481334	6.62	502235	7.29	22
39	481731	6.62	502672	7.28	21
40	482128	6.61	503109	7.28	20
41	9.482525	6.60	9.503546	7.27	19
42	482921	6.59	503982	7.27	18
43	483316	6.59	504418	7.26	17
44	483712	6.58	504854	7.25	16
45	484107	6.57	505289	7.25	15
46	484501	6.57	505724	7.24	14
47	484895	6.56	506159	7.24	13
48	485289	6.55	506593	7.23	12
49	485682	6.55	507027	7.22	11
50	486075	6.54	507460	7.22	10
51	9.486467	6.53	9.507893	7.22	9
52	486860	6.53	508326	7.21	8
53	487251	6.52	508759	7.20	7
54	487643	6.51	509191	7.19	6
55	488034	6.50	509622	7.19	5
56	488424	6.50	510054	7.18	4
57	488814	6.50	510485	7.18	3
58	489204	6.49	510916	7.17	2
59	489593	6.48	511346	7.17	1
60	489982		511776		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.489982	6.48	9.511776	7.16	60	0	9.512642	6.12	9.536972	6.84	60
1	490371	6.47	512206	7.16	59	1	513009	6.11	537382	6.83	59
2	490759	6.47	512635	7.15	58	2	513375	6.10	537792	6.83	58
3	491147	6.46	513064	7.14	57	3	513741	6.10	538202	6.82	57
4	491535	6.45	513493	7.14	56	4	514107	6.09	538611	6.82	56
5	491922	6.45	513921	7.13	55	5	514472	6.08	539020	6.81	55
6	492308	6.44	514349	7.13	54	6	514837	6.08	539429	6.81	54
7	492695	6.43	514777	7.12	53	7	515202	6.07	539837	6.80	53
8	493081	6.43	515204	7.12	52	8	515566	6.07	540245	6.80	52
9	493466	6.42	515631	7.11	51	9	515930	6.06	540653	6.79	51
10	493851	6.42	516057	7.10	50	10	516294	6.05	541061	6.79	50
11	9.494236	6.41	9.516484	7.10	49	11	9.516657	6.05	9.541468	6.78	49
12	494621	6.40	516910	7.09	48	12	517020	6.04	541875	6.78	48
13	495005	6.40	517335	7.09	47	13	517382	6.04	542281	6.77	47
14	495388	6.39	517761	7.08	46	14	517745	6.03	542688	6.77	46
15	495772	6.38	518186	7.08	45	15	518107	6.02	543094	6.76	45
16	496154	6.38	518610	7.07	44	16	518468	6.03	543499	6.76	44
17	496537	6.37	519034	7.06	43	17	518829	6.02	543905	6.75	43
18	496919	6.37	519458	7.06	42	18	519190	6.01	544310	6.75	42
19	497301	6.36	519882	7.05	41	19	519551	6.00	544715	6.74	41
20	497682	6.36	520305	7.05	40	20	519911	6.00	545119	6.74	40
21	9.498064	6.35	9.520728	7.04	39	21	9.520271	6.00	9.545524	6.73	39
22	498444	6.34	521151	7.03	38	22	520631	5.99	545928	6.73	38
23	498825	6.33	521573	7.03	37	23	520990	5.98	546331	6.72	37
24	499204	6.33	521995	7.03	36	24	521349	5.97	546735	6.72	36
25	499584	6.32	522417	7.02	35	25	521707	5.98	547138	6.71	35
26	499963	6.32	522838	7.02	34	26	522066	5.97	547540	6.71	34
27	500342	6.31	523259	7.01	33	27	522424	5.96	547943	6.70	33
28	500721	6.30	523680	7.01	32	28	522781	5.95	548345	6.70	32
29	501099	6.30	524100	7.00	31	29	523138	5.95	548747	6.69	31
30	501476	6.29	524520	6.99	30	30	523495	5.95	549149	6.69	30
31	9.501854	6.28	9.524940	6.99	29	31	9.523852	5.94	9.549550	6.68	29
32	502231	6.28	525359	6.98	28	32	524208	5.93	549951	6.68	28
33	502607	6.28	525778	6.98	27	33	524564	5.93	550352	6.67	27
34	502984	6.27	526197	6.97	26	34	524920	5.92	550752	6.67	26
35	503360	6.26	526615	6.97	25	35	525275	5.92	551153	6.66	25
36	503735	6.26	527033	6.96	24	36	525630	5.90	551552	6.66	24
37	504110	6.25	527451	6.95	23	37	525984	5.91	551952	6.65	23
38	504485	6.25	527868	6.95	22	38	526339	5.90	552351	6.65	22
39	504860	6.24	528285	6.95	21	39	526693	5.89	552750	6.65	21
40	505234	6.23	528702	6.94	20	40	527046	5.90	553149	6.64	20
41	9.505608	6.22	9.529119	6.93	19	41	9.527400	5.89	9.553548	6.64	19
42	505981	6.22	529535	6.93	18	42	527753	5.87	553946	6.63	18
43	506354	6.22	529951	6.93	17	43	528105	5.88	554344	6.63	17
44	506727	6.20	530366	6.92	16	44	528458	5.87	554741	6.62	16
45	507099	6.20	530781	6.91	15	45	528810	5.85	555139	6.62	15
46	507471	6.20	531196	6.91	14	46	529161	5.86	555536	6.61	14
47	507843	6.19	531611	6.90	13	47	529513	5.85	555933	6.61	13
48	508214	6.18	532025	6.90	12	48	529864	5.85	556329	6.60	12
49	508585	6.18	532439	6.89	11	49	530215	5.84	556725	6.60	11
50	508956	6.17	532853	6.89	10	50	530565	5.83	557121	6.59	10
51	9.509326	6.17	9.533266	6.88	9	51	9.530915	5.83	9.557517	6.59	9
52	509693	6.16	533679	6.88	8	52	531265	5.82	557913	6.59	8
53	510063	6.15	534092	6.87	7	53	531614	5.82	558308	6.58	7
54	510431	6.15	534504	6.87	6	54	531963	5.82	558703	6.58	6
55	510803	6.15	534916	6.86	5	55	532312	5.81	559097	6.57	5
56	511172	6.14	535328	6.86	4	56	532661	5.80	559491	6.57	4
57	511540	6.13	535739	6.85	3	57	533009	5.80	559885	6.56	3
58	511907	6.13	536150	6.85	2	58	533357	5.80	560279	6.56	2
59	512275	6.12	536561	6.84	1	59	533704	5.79	560673	6.55	1
60	512642		536972		0	60	534052		561066		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

20° TABLE IV.—LOGARITHMIC 21°

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.534052	5.78	9.561066	6.55	60
1	534399	5.77	561459	6.54	59
2	534745	5.77	561851	6.54	58
3	535092	5.77	562244	6.53	57
4	535438	5.76	562636	6.53	56
5	535783	5.76	563028	6.53	55
6	536129	5.75	563419	6.52	54
7	536474	5.74	563811	6.52	53
8	536818	5.74	564202	6.51	52
9	537163	5.73	564593	6.51	51
10	537507	5.73	564983	6.50	50
11	9.537851	5.72	9.565373	6.50	49
12	538194	5.72	565763	6.49	48
13	538538	5.71	566153	6.49	47
14	538880	5.71	566542	6.49	46
15	539223	5.70	566932	6.48	45
16	539565	5.70	567320	6.48	44
17	539907	5.70	567709	6.47	43
18	540249	5.69	568098	6.47	42
19	540590	5.68	568486	6.46	41
20	540931	5.68	568873	6.46	40
21	9.541272	5.67	9.569261	6.45	39
22	541613	5.67	569648	6.45	38
23	541953	5.66	570035	6.45	37
24	542293	5.65	570422	6.44	36
25	542632	5.65	570809	6.44	35
26	542971	5.65	571195	6.43	34
27	543310	5.65	571581	6.43	33
28	543649	5.64	571967	6.42	32
29	543987	5.63	572352	6.42	31
30	544325	5.63	572738	6.42	30
31	9.544663	5.62	9.573123	6.41	29
32	545000	5.62	573507	6.41	28
33	545338	5.61	573892	6.40	27
34	545674	5.61	574276	6.40	26
35	546011	5.60	574660	6.39	25
36	546347	5.60	575044	6.39	24
37	546683	5.60	575427	6.39	23
38	547019	5.59	575810	6.38	22
39	547354	5.58	576193	6.38	21
40	547689	5.58	576576	6.37	20
41	9.548024	5.57	9.576959	6.37	19
42	548359	5.57	577341	6.36	18
43	548693	5.56	577723	6.36	17
44	549027	5.55	578104	6.36	16
45	549360	5.55	578486	6.35	15
46	549693	5.55	578867	6.35	14
47	550026	5.55	579248	6.34	13
48	550359	5.54	579629	6.34	12
49	550692	5.53	580009	6.34	11
50	551024	5.53	580389	6.33	10
51	9.551356	5.52	9.580769	6.33	9
52	551687	5.52	581149	6.32	8
53	552018	5.52	581528	6.32	7
54	552349	5.51	581907	6.32	6
55	552680	5.50	582286	6.31	5
56	553010	5.50	582665	6.31	4
57	553341	5.50	583044	6.30	3
58	553670	5.50	583422	6.30	2
59	554000	5.49	583800	6.29	1
60	554329		584177		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.554329	5.48	9.584177	6.29	60
1	554658	5.48	584555	6.29	59
2	554987	5.47	584932	6.28	58
3	555315	5.47	585309	6.28	57
4	555643	5.47	585686	6.27	56
5	555971	5.46	586062	6.27	55
6	556299	5.45	586439	6.27	54
7	556626	5.45	586815	6.26	53
8	556953	5.45	587190	6.26	52
9	557280	5.44	587566	6.25	51
10	557606	5.43	587941	6.25	50
11	9.557932	5.43	9.588316	6.25	49
12	558258	5.42	588691	6.24	48
13	558583	5.42	589066	6.24	47
14	558909	5.42	589440	6.23	46
15	559234	5.41	589814	6.23	45
16	559558	5.41	590188	6.23	44
17	559883	5.40	590562	6.22	43
18	560207	5.40	590935	6.22	42
19	560531	5.40	591308	6.22	41
20	560855	5.39	591681	6.21	40
21	9.561178	5.38	9.592054	6.21	39
22	561501	5.38	592426	6.20	38
23	561824	5.37	592799	6.20	37
24	562146	5.37	593171	6.19	36
25	562468	5.37	593542	6.19	35
26	562790	5.36	593914	6.18	34
27	563112	5.35	594285	6.18	33
28	563433	5.35	594656	6.18	32
29	563755	5.35	595027	6.18	31
30	564075	5.34	595398	6.17	30
31	9.564396	5.34	9.595768	6.17	29
32	564716	5.33	596138	6.16	28
33	565036	5.33	596508	6.16	27
34	565356	5.33	596878	6.16	26
35	565676	5.32	597247	6.15	25
36	565995	5.32	597616	6.15	24
37	566314	5.31	597985	6.15	23
38	566632	5.31	598354	6.14	22
39	566951	5.30	598722	6.14	21
40	567269	5.30	599091	6.13	20
41	9.567587	5.29	9.599459	6.13	19
42	567904	5.29	599827	6.13	18
43	568222	5.28	600194	6.12	17
44	568539	5.28	600562	6.12	16
45	568856	5.27	600929	6.11	15
46	569172	5.27	601296	6.11	14
47	569488	5.27	601663	6.11	13
48	569804	5.26	602029	6.10	12
49	570120	5.25	602395	6.10	11
50	570435	5.25	602761	6.10	10
51	9.570751	5.25	9.603127	6.09	9
52	571066	5.24	603493	6.09	8
53	571380	5.24	603858	6.09	7
54	571695	5.23	604223	6.08	6
55	572009	5.23	604588	6.08	5
56	572323	5.23	604953	6.07	4
57	572636	5.22	605317	6.07	3
58	572950	5.21	605682	6.07	2
59	573263	5.20	606046	6.06	1
60	573575		606410		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

22°

M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.573575	5.21	9.606410	6.06	60
1	573888	5.20	606773	6.06	59
2	574200	5.20	607137	6.05	58
3	574512	5.20	607500	6.05	57
4	574824	5.20	607863	6.04	56
5	575136	5.19	608225	6.04	55
6	575447	5.18	608588	6.04	54
7	575758	5.18	608950	6.03	53
8	576069	5.17	609312	6.03	52
9	576379	5.17	609674	6.03	51
10	576689	5.17	610036	6.02	50
11	9.576999	5.16	9.610397	6.02	49
12	577309	5.15	610759	6.02	48
13	577618	5.15	611120	6.01	47
14	577927	5.15	611480	6.01	46
15	578236	5.15	611841	6.01	45
16	578545	5.14	612201	6.00	44
17	578853	5.13	612561	6.00	43
18	579162	5.13	612921	6.00	42
19	579470	5.13	613281	5.99	41
20	579777	5.12	613641	5.99	40
21	9.580085	5.12	9.614000	5.98	39
22	580392	5.12	614359	5.98	38
23	580699	5.11	614718	5.98	37
24	581005	5.11	615077	5.97	36
25	581312	5.10	615435	5.97	35
26	581618	5.10	615793	5.97	34
27	581924	5.09	616151	5.96	33
28	582229	5.09	616509	5.96	32
29	582535	5.08	616867	5.96	31
30	582840	5.08	617224	5.95	30
31	9.583145	5.08	9.617582	5.95	29
32	583449	5.07	617939	5.95	28
33	583754	5.07	618295	5.94	27
34	584058	5.06	618652	5.94	26
35	584361	5.06	619008	5.94	25
36	584665	5.05	619364	5.93	24
37	584968	5.05	619720	5.93	23
38	585272	5.05	620076	5.93	22
39	585574	5.04	620432	5.92	21
40	585877	5.04	620787	5.92	20
41	9.586179	5.03	9.621142	5.92	19
42	586482	5.03	621497	5.91	18
43	586783	5.03	621852	5.91	17
44	587085	5.02	622207	5.90	16
45	587386	5.02	622561	5.90	15
46	587688	5.01	622915	5.90	14
47	587989	5.01	623269	5.89	13
48	588289	5.01	623623	5.89	12
49	588590	5.00	623976	5.89	11
50	588890	5.00	624330	5.88	10
51	9.589190	4.99	9.624683	5.88	9
52	589489	4.99	625036	5.88	8
53	589789	4.98	625388	5.87	7
54	590088	4.98	625741	5.87	6
55	590387	4.98	626093	5.87	5
56	590586	4.97	626445	5.86	4
57	590984	4.97	626797	5.86	3
58	591282	4.97	627149	5.86	2
59	591580	4.97	627501	5.85	1
60	591878		627852		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.

67°

23°

M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.591878	4.96	9.627852	5.85	60
1	592176	4.95	628203	5.85	59
2	592473	4.95	628554	5.85	58
3	592770	4.95	628905	5.84	57
4	593067	4.94	629255	5.84	56
5	593363	4.94	629606	5.83	55
6	593659	4.93	629956	5.83	54
7	593955	4.93	630306	5.83	53
8	594251	4.93	630656	5.83	52
9	594547	4.92	631005	5.82	51
10	594842	4.92	631355	5.82	50
11	9.595137	4.92	9.631704	5.82	49
12	595432	4.91	632053	5.81	48
13	595727	4.90	632402	5.81	47
14	596021	4.90	632750	5.81	46
15	596315	4.90	633099	5.80	45
16	596609	4.90	633447	5.80	44
17	596903	4.89	633795	5.80	43
18	597196	4.89	634143	5.79	42
19	597490	4.88	634490	5.79	41
20	597783	4.88	634838	5.79	40
21	9.598075	4.87	9.635185	5.78	39
22	598368	4.87	635532	5.78	38
23	598660	4.87	635879	5.78	37
24	598952	4.86	636226	5.77	36
25	599244	4.86	636572	5.77	35
26	599536	4.85	636919	5.77	34
27	599827	4.85	637265	5.77	33
28	600118	4.85	637611	5.76	32
29	600409	4.85	637956	5.76	31
30	600700	4.84	638302	5.76	30
31	600990	4.83	9.638647	5.75	29
32	601280	4.83	638992	5.75	28
33	601570	4.83	639337	5.75	27
34	601860	4.83	639682	5.74	26
35	602150	4.82	640027	5.74	25
36	602439	4.82	640371	5.74	24
37	602728	4.82	640716	5.73	23
38	603017	4.81	641060	5.73	22
39	603305	4.81	641404	5.73	21
40	603594	4.80	641747	5.72	20
41	9.603882	4.80	9.642091	5.72	19
42	604170	4.79	642434	5.72	18
43	604457	4.79	642777	5.72	17
44	604745	4.79	643120	5.71	16
45	605032	4.78	643463	5.71	15
46	605319	4.78	643806	5.71	14
47	605606	4.78	644148	5.70	13
48	605892	4.77	644490	5.70	12
49	606179	4.77	644832	5.70	11
50	606465	4.76	645174	5.69	10
51	9.606751	4.76	9.645516	5.69	9
52	607036	4.76	645857	5.69	8
53	607322	4.75	646199	5.69	7
54	607607	4.75	646540	5.68	6
55	607892	4.75	646881	5.68	5
56	608177	4.74	647222	5.68	4
57	608461	4.74	647562	5.67	3
58	608745	4.73	647903	5.67	2
59	609029	4.73	648243	5.67	1
60	609313		648583		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.

66°

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.609313	4.73	9.648583	5.66	60	0	9.625948	4.51	9.668673	5.50	60
1	609597	4.72	648923	5.66	59	1	626219	4.51	669002	5.49	59
2	609880	4.72	649263	5.66	58	2	626490	4.50	669332	5.49	58
3	610164	4.72	649602	5.66	57	3	626760	4.50	669661	5.49	57
4	610447	4.71	649942	5.65	56	4	627030	4.50	669991	5.48	56
5	610729	4.71	650281	5.65	55	5	627300	4.50	670320	5.48	55
6	611012	4.70	650620	5.65	54	6	627570	4.50	670649	5.48	54
7	611294	4.70	650959	5.64	53	7	627840	4.49	670977	5.48	53
8	611576	4.70	651297	5.64	52	8	628109	4.49	671306	5.47	52
9	611858	4.70	651636	5.64	51	9	628378	4.48	671635	5.47	51
10	612140	4.69	651974	5.63	50	10	628647	4.48	671963	5.47	50
11	9.612421	4.69	9.652312	5.63	49	11	9.628916	4.47	9.672291	5.47	49
12	612702	4.68	652650	5.63	48	12	629185	4.47	672619	5.46	48
13	612983	4.68	652988	5.63	47	13	629453	4.47	672947	5.46	47
14	613264	4.67	653326	5.62	46	14	629721	4.46	673274	5.46	46
15	613545	4.67	653663	5.62	45	15	629989	4.46	673602	5.46	45
16	613825	4.67	654000	5.62	44	16	630257	4.46	673929	5.45	44
17	614105	4.66	654337	5.61	43	17	630524	4.46	674257	5.45	43
18	614385	4.66	654674	5.61	42	18	630792	4.45	674584	5.45	42
19	614665	4.66	655011	5.61	41	19	631059	4.45	674911	5.44	41
20	614944	4.65	655348	5.61	40	20	631326	4.45	675237	5.44	40
21	9.615223	4.65	9.655684	5.60	39	21	9.631593	4.44	9.675564	5.44	39
22	615502	4.65	656020	5.60	38	22	631859	4.44	675890	5.44	38
23	615781	4.64	656356	5.60	37	23	632125	4.44	676217	5.43	37
24	616030	4.64	656692	5.59	36	24	632392	4.43	676543	5.43	36
25	616338	4.64	657028	5.59	35	25	632658	4.43	676869	5.43	35
26	616616	4.63	657364	5.59	34	26	632923	4.43	677194	5.43	34
27	616894	4.63	657699	5.59	33	27	633189	4.42	677520	5.42	33
28	617172	4.62	658034	5.58	32	28	633454	4.42	677846	5.42	32
29	617450	4.62	658369	5.58	31	29	633719	4.42	678171	5.42	31
30	617727	4.62	658704	5.58	30	30	633984	4.41	678496	5.42	30
31	9.618004	4.61	9.659039	5.58	29	31	9.634249	4.41	9.678821	5.41	29
32	618281	4.61	659373	5.57	28	32	634514	4.40	679146	5.41	28
33	618558	4.61	659708	5.57	27	33	634778	4.40	679471	5.41	27
34	618834	4.60	660042	5.57	26	34	635042	4.40	679795	5.41	26
35	619110	4.60	660376	5.57	25	35	635306	4.40	680120	5.40	25
36	619386	4.60	660710	5.56	24	36	635570	4.39	680444	5.40	24
37	619662	4.59	661043	5.56	23	37	635834	4.39	680768	5.40	23
38	619938	4.59	661377	5.56	22	38	636097	4.38	681092	5.40	22
39	620213	4.59	661710	5.55	21	39	636360	4.38	681416	5.39	21
40	620488	4.58	662043	5.55	20	40	636623	4.38	681740	5.39	20
41	9.620763	4.58	9.662376	5.55	19	41	9.636886	4.37	9.682063	5.39	19
42	621038	4.57	662709	5.54	18	42	637148	4.37	682387	5.39	18
43	621313	4.57	663042	5.54	17	43	637411	4.37	682710	5.38	17
44	621587	4.57	663375	5.54	16	44	637673	4.37	683033	5.38	16
45	621861	4.56	663707	5.54	15	45	637935	4.36	683356	5.38	15
46	622135	4.56	664039	5.53	14	46	638197	4.36	683679	5.38	14
47	622409	4.56	664371	5.53	13	47	638458	4.36	684001	5.37	13
48	622682	4.55	664703	5.53	12	48	638720	4.35	684324	5.37	12
49	622956	4.55	665035	5.53	11	49	638981	4.35	684646	5.37	11
50	623229	4.55	665366	5.52	10	50	639242	4.35	684968	5.37	10
51	9.623502	4.54	9.665698	5.52	9	51	9.639503	4.34	9.685290	5.36	9
52	623774	4.54	666029	5.52	8	52	639764	4.34	685612	5.36	8
53	624047	4.54	666360	5.51	7	53	640024	4.34	685934	5.36	7
54	624319	4.53	666691	5.51	6	54	640284	4.33	686255	5.36	6
55	624591	4.53	667021	5.51	5	55	640544	4.33	686577	5.35	5
56	624863	4.53	667352	5.51	4	56	640804	4.33	686898	5.35	4
57	625135	4.52	667682	5.50	3	57	641064	4.32	687219	5.35	3
58	625406	4.52	668013	5.50	2	58	641324	4.32	687540	5.35	2
59	625677	4.52	668343	5.50	1	59	641583	4.32	687861	5.34	1
60	625948		668673		0	60	641842		688182		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.641842	4.31	9.688182	5.34	60
1	642101	4.31	688502	5.34	59
2	642360	4.30	688823	5.34	58
3	642618	4.30	689143	5.33	57
4	642877	4.30	689463	5.33	56
5	643135	4.30	689783	5.33	55
6	643393	4.30	690103	5.33	54
7	643650	4.29	690423	5.33	53
8	643908	4.29	690742	5.32	52
9	644165	4.29	691062	5.32	51
10	644423	4.28	691381	5.32	50
11	9.644680	4.28	9.691700	5.31	49
12	644936	4.28	692019	5.31	48
13	645193	4.27	692338	5.31	47
14	645450	4.27	692656	5.31	46
15	645706	4.27	692975	5.31	45
16	645962	4.26	693293	5.30	44
17	646218	4.26	693612	5.30	43
18	646474	4.26	693930	5.30	42
19	646729	4.25	694248	5.30	41
20	646984	4.25	694566	5.29	40
21	9.647240	4.24	9.694883	5.29	39
22	647494	4.24	695201	5.29	38
23	647749	4.24	695518	5.29	37
24	648004	4.23	695836	5.29	36
25	648258	4.23	696153	5.28	35
26	648512	4.23	696470	5.28	34
27	648766	4.23	696787	5.28	33
28	649020	4.23	697103	5.28	32
29	649274	4.22	697420	5.27	31
30	649527	4.22	697736	5.27	30
31	9.649781	4.22	9.698053	5.27	29
32	650034	4.22	698369	5.27	28
33	650287	4.21	698685	5.26	27
34	650539	4.21	699001	5.26	26
35	650792	4.21	699316	5.26	25
36	651044	4.20	699632	5.26	24
37	651297	4.20	699947	5.26	23
38	651549	4.20	700263	5.25	22
39	651800	4.19	700578	5.25	21
40	652052	4.19	700893	5.25	20
41	9.652304	4.18	9.701208	5.24	19
42	652555	4.18	701523	5.24	18
43	652806	4.18	701837	5.24	17
44	653057	4.18	702152	5.24	16
45	653308	4.17	702466	5.24	15
46	653558	4.17	702781	5.23	14
47	653808	4.17	703095	5.23	13
48	654059	4.17	703409	5.23	12
49	654309	4.16	703722	5.23	11
50	654558	4.16	704036	5.22	10
51	9.654808	4.16	9.704350	5.22	9
52	655058	4.16	704663	5.22	8
53	655307	4.15	704976	5.22	7
54	655556	4.15	705290	5.22	6
55	655805	4.15	705603	5.21	5
56	656054	4.14	705916	5.21	4
57	656302	4.14	706228	5.21	3
58	656551	4.14	706541	5.21	2
59	656799	4.13	706854	5.21	1
60	657047		707166		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.657047	4.13	9.707166	5.20	60
1	657295	4.13	707478	5.20	59
2	657542	4.12	707790	5.20	58
3	657790	4.12	708102	5.20	57
4	658037	4.12	708414	5.19	56
5	658284	4.12	708726	5.19	55
6	658531	4.11	709037	5.19	54
7	658778	4.11	709349	5.19	53
8	659025	4.11	709660	5.19	52
9	659271	4.10	709971	5.18	51
10	659517	4.10	710282	5.18	50
11	9.659763	4.10	9.710593	5.18	49
12	660009	4.09	710904	5.18	48
13	660255	4.09	711215	5.18	47
14	660501	4.09	711525	5.17	46
15	660746	4.09	711836	5.17	45
16	660991	4.08	712146	5.17	44
17	661236	4.08	712456	5.17	43
18	661481	4.08	712766	5.16	42
19	661726	4.07	713076	5.16	41
20	661970	4.07	713386	5.16	40
21	9.662214	4.07	9.713696	5.16	39
22	662459	4.07	714005	5.16	38
23	662703	4.06	714314	5.15	37
24	662946	4.06	714624	5.15	36
25	663190	4.06	714933	5.15	35
26	663433	4.05	715242	5.15	34
27	663677	4.05	715551	5.14	33
28	663920	4.05	715860	5.14	32
29	664163	4.05	716168	5.14	31
30	664406	4.04	716477	5.14	30
31	9.664648	4.04	9.716785	5.14	29
32	664891	4.04	717093	5.13	28
33	665133	4.03	717401	5.13	27
34	665375	4.03	717709	5.13	26
35	665617	4.03	718017	5.13	25
36	665859	4.02	718325	5.13	24
37	666100	4.02	718633	5.12	23
38	666342	4.02	718940	5.12	22
39	666583	4.02	719248	5.12	21
40	666824	4.01	719555	5.12	20
41	9.667065	4.01	9.719862	5.12	19
42	667305	4.01	720169	5.11	18
43	667546	4.01	720476	5.11	17
44	667786	4.00	720783	5.11	16
45	668027	4.00	721089	5.11	15
46	668267	4.00	721396	5.11	14
47	668506	3.99	721702	5.10	13
48	668746	3.99	722009	5.10	12
49	668986	3.99	722315	5.10	11
50	669225	3.99	722621	5.10	10
51	9.669464	3.98	9.722927	5.10	9
52	669703	3.98	723232	5.09	8
53	669942	3.98	723538	5.09	7
54	670181	3.97	723844	5.09	6
55	670419	3.97	724149	5.09	5
56	670658	3.97	724454	5.09	4
57	670896	3.97	724760	5.08	3
58	671134	3.96	725065	5.08	2
59	671372	3.96	725370	5.08	1
60	671609		725674		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.671609	3.96	9.725674	5.08	60	0	9.685571	3.80	9.743752	4.96	60
1	671847	3.95	725979	5.08	59	1	685799	3.79	744050	4.96	59
2	672084	3.95	726284	5.07	58	2	686027	3.79	744348	4.96	58
3	672321	3.95	726588	5.07	57	3	686254	3.79	744645	4.96	57
4	672558	3.95	726892	5.07	56	4	686482	3.79	744943	4.96	56
5	672795	3.94	727197	5.07	55	5	686709	3.78	745240	4.96	55
6	673032	3.94	727501	5.07	54	6	686936	3.78	745538	4.95	54
7	673268	3.94	727805	5.06	53	7	687163	3.78	745835	4.95	53
8	673505	3.94	728109	5.06	52	8	687389	3.78	746132	4.95	52
9	673741	3.93	728412	5.06	51	9	687616	3.77	746429	4.95	51
10	673977	3.93	728716	5.06	50	10	687843	3.77	746726	4.95	50
11	9.674213	3.93	9.729020	5.06	49	11	9.688069	3.77	9.747023	4.94	49
12	674448	3.92	729323	5.05	48	12	688295	3.77	747319	4.94	48
13	674684	3.92	729626	5.05	47	13	688521	3.76	747616	4.94	47
14	674919	3.92	729929	5.05	46	14	688747	3.76	747913	4.94	46
15	675155	3.92	730233	5.05	45	15	688972	3.76	748209	4.94	45
16	675390	3.91	730535	5.05	44	16	689198	3.76	748505	4.93	44
17	675624	3.91	730838	5.04	43	17	689423	3.75	748801	4.93	43
18	675859	3.91	731141	5.04	42	18	689648	3.75	749097	4.93	42
19	676094	3.91	731444	5.04	41	19	689873	3.75	749393	4.93	41
20	676328	3.90	731746	5.04	40	20	690098	3.75	749689	4.93	40
21	9.676562	3.90	9.732048	5.04	39	21	9.690323	3.74	9.749985	4.93	39
22	676796	3.90	732351	5.03	38	22	690548	3.74	750281	4.92	38
23	677030	3.90	732653	5.03	37	23	690772	3.74	750576	4.92	37
24	677264	3.89	732955	5.03	36	24	690996	3.74	750872	4.92	36
25	677498	3.89	733257	5.03	35	25	691220	3.73	751167	4.92	35
26	677731	3.89	733558	5.03	34	26	691444	3.73	751462	4.92	34
27	677964	3.88	733860	5.02	33	27	691668	3.73	751757	4.92	33
28	678197	3.88	734162	5.02	32	28	691892	3.73	752052	4.91	32
29	678430	3.88	734463	5.02	31	29	692115	3.72	752347	4.91	31
30	678663	3.88	734764	5.02	30	30	692339	3.72	752642	4.91	30
31	9.678895	3.87	9.735066	5.02	29	31	9.692562	3.72	9.752937	4.91	29
32	679128	3.87	735367	5.02	28	32	692785	3.71	753231	4.91	28
33	679360	3.87	735668	5.01	27	33	693008	3.71	753526	4.91	27
34	679592	3.87	735969	5.01	26	34	693231	3.71	753820	4.90	26
35	679824	3.86	736269	5.01	25	35	693453	3.71	754115	4.90	25
36	680056	3.86	736570	5.01	24	36	693676	3.70	754409	4.90	24
37	680288	3.86	736870	5.01	23	37	693898	3.70	754703	4.90	23
38	680519	3.85	737171	5.00	22	38	694120	3.70	754997	4.90	22
39	680750	3.85	737471	5.00	21	39	694342	3.70	755291	4.90	21
40	680982	3.85	737771	5.00	20	40	694564	3.69	755585	4.89	20
41	9.681213	3.85	9.738071	5.00	19	41	9.694786	3.69	9.755878	4.89	19
42	681443	3.84	738371	5.00	18	42	695007	3.69	756172	4.89	18
43	681674	3.84	738671	4.99	17	43	695229	3.69	756465	4.89	17
44	681905	3.84	738971	4.99	16	44	695450	3.68	756759	4.89	16
45	682135	3.84	739271	4.99	15	45	695671	3.68	757052	4.89	15
46	682365	3.83	739570	4.99	14	46	695892	3.68	757345	4.88	14
47	682595	3.83	739870	4.99	13	47	696113	3.68	757638	4.88	13
48	682825	3.83	740169	4.99	12	48	696334	3.67	757931	4.88	12
49	683055	3.83	740468	4.98	11	49	696554	3.67	758224	4.88	11
50	683284	3.82	740767	4.98	10	50	696775	3.67	758517	4.88	10
51	9.683514	3.82	9.741066	4.98	9	51	9.696995	3.67	9.758810	4.88	9
52	683743	3.82	741365	4.98	8	52	697215	3.66	759102	4.87	8
53	683972	3.82	741664	4.98	7	53	697435	3.66	759395	4.87	7
54	684201	3.81	741962	4.97	6	54	697654	3.66	759687	4.87	6
55	684430	3.81	742261	4.97	5	55	697874	3.66	759979	4.87	5
56	684658	3.81	742559	4.97	4	56	698094	3.65	760272	4.87	4
57	684887	3.80	742858	4.97	3	57	698313	3.65	760564	4.87	3
58	685115	3.80	743156	4.97	2	58	698532	3.65	760856	4.86	2
59	685343	3.80	743454	4.97	1	59	698751	3.65	761148	4.86	1
60	685571		743752		0	60	698970		761439		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.698970	3.65	9.761439	4.86	60
1	699189	3.64	761731	4.86	59
2	699407	3.64	762023	4.86	58
3	699626	3.64	762314	4.86	57
4	699844	3.63	762606	4.85	56
5	700062	3.63	762897	4.85	55
6	700280	3.63	763188	4.85	54
7	700498	3.63	763479	4.85	53
8	700716	3.63	763770	4.85	52
9	700933	3.62	764061	4.85	51
10	701151	3.62	764352	4.85	50
11	9.701368	3.62	9.764643	4.84	49
12	701585	3.62	764933	4.84	48
13	701802	3.62	765224	4.84	47
14	702019	3.62	765514	4.84	46
15	702236	3.61	765805	4.84	45
16	702452	3.61	766095	4.83	44
17	702669	3.60	766385	4.83	43
18	702885	3.60	766675	4.83	42
19	703101	3.60	766965	4.83	41
20	703317	3.60	767255	4.83	40
21	9.703533	3.60	9.767545	4.83	39
22	703749	3.59	767834	4.83	38
23	703964	3.59	768124	4.82	37
24	704179	3.59	768414	4.82	36
25	704395	3.58	768703	4.82	35
26	704610	3.58	768992	4.82	34
27	704825	3.58	769281	4.82	33
28	705040	3.58	769571	4.82	32
29	705254	3.58	769860	4.81	31
30	705469	3.57	770148	4.81	30
31	9.705683	3.57	9.770437	4.81	29
32	705898	3.57	770726	4.81	28
33	706112	3.57	771015	4.81	27
34	706326	3.56	771303	4.81	26
35	706539	3.56	771592	4.80	25
36	706753	3.56	771880	4.80	24
37	706967	3.55	772168	4.80	23
38	707180	3.55	772457	4.80	22
39	707393	3.55	772745	4.80	21
40	707606	3.55	773033	4.80	20
41	9.707819	3.55	9.773321	4.80	19
42	708032	3.55	773608	4.79	18
43	708245	3.54	773896	4.79	17
44	708458	3.54	774184	4.79	16
45	708670	3.53	774471	4.79	15
46	708882	3.53	774759	4.79	14
47	709094	3.53	775046	4.79	13
48	709306	3.53	775333	4.79	12
49	709518	3.53	775621	4.78	11
50	709730	3.52	775908	4.78	10
51	9.709941	3.52	9.776195	4.78	9
52	710153	3.52	776482	4.78	8
53	710364	3.52	776768	4.78	7
54	710575	3.52	777055	4.78	6
55	710786	3.52	777342	4.78	5
56	710997	3.51	777628	4.77	4
57	711208	3.51	777915	4.77	3
58	711419	3.50	778201	4.77	2
59	711629	3.50	778488	4.77	1
60	711839		778774		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.711839	3.50	9.778774	4.77	60
1	712050	3.50	779060	4.77	59
2	712260	3.50	779346	4.77	58
3	712469	3.50	779632	4.76	57
4	712679	3.50	779918	4.76	56
5	712889	3.49	780203	4.76	55
6	713098	3.49	780489	4.76	54
7	713308	3.49	780775	4.76	53
8	713517	3.48	781060	4.76	52
9	713726	3.48	781346	4.75	51
10	713935	3.48	781631	4.75	50
11	9.714144	3.48	9.781916	4.75	49
12	714352	3.47	782201	4.75	48
13	714561	3.47	782486	4.75	47
14	714769	3.47	782771	4.75	46
15	714978	3.47	783056	4.75	45
16	715186	3.47	783341	4.75	44
17	715394	3.46	783626	4.74	43
18	715602	3.46	783910	4.74	42
19	715809	3.46	784195	4.74	41
20	716017	3.46	784479	4.74	40
21	9.716224	3.45	9.784764	4.74	39
22	716432	3.45	785048	4.73	38
23	716639	3.45	785332	4.73	37
24	716846	3.45	785616	4.73	36
25	717053	3.45	785900	4.73	35
26	717259	3.44	786184	4.73	34
27	717466	3.44	786468	4.73	33
28	717673	3.44	786752	4.73	32
29	717879	3.43	787036	4.73	31
30	718085	3.43	787319	4.72	30
31	9.718291	3.43	9.787603	4.72	29
32	718497	3.43	787886	4.72	28
33	718703	3.43	788170	4.72	27
34	718909	3.43	788453	4.72	26
35	719114	3.42	788736	4.72	25
36	719320	3.42	789019	4.72	24
37	719525	3.42	789302	4.72	23
38	719730	3.42	789585	4.71	22
39	719935	3.41	789868	4.71	21
40	720140	3.41	790151	4.71	20
41	9.720345	3.41	9.790434	4.71	19
42	720549	3.41	790716	4.71	18
43	720754	3.40	790999	4.71	17
44	720958	3.40	791281	4.70	16
45	721162	3.40	791563	4.70	15
46	721366	3.40	791846	4.70	14
47	721570	3.40	792128	4.70	13
48	721774	3.40	792410	4.70	12
49	721978	3.39	792692	4.70	11
50	722181	3.39	792974	4.70	10
51	9.722385	3.39	9.793256	4.70	9
52	722588	3.38	793538	4.69	8
53	722791	3.38	793819	4.69	7
54	722994	3.38	794101	4.69	6
55	723197	3.38	794383	4.69	5
56	723400	3.38	794664	4.69	4
57	723603	3.37	794946	4.69	3
58	723805	3.37	795227	4.69	2
59	724007	3.37	795508	4.68	1
60	724210		795789		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.724210	3.37	9.795789	4.68	60	0	9.736109	3.24	9.812517	4.61	60
1	724412	3.37	796070	4.68	59	1	736303	3.24	812794	4.61	59
2	724614	3.37	796351	4.68	58	2	736498	3.24	813070	4.61	58
3	724816	3.36	796632	4.68	57	3	736692	3.23	813347	4.60	57
4	725017	3.36	796913	4.68	56	4	736886	3.23	813623	4.60	56
5	725219	3.36	797194	4.68	55	5	737080	3.23	813899	4.60	55
6	725420	3.35	797474	4.68	54	6	737274	3.23	814176	4.60	54
7	725622	3.35	797755	4.68	53	7	737467	3.23	814452	4.60	53
8	725823	3.35	798036	4.67	52	8	737661	3.22	814728	4.60	52
9	726024	3.35	798316	4.67	51	9	737855	3.22	815004	4.60	51
10	726225	3.35	798596	4.67	50	10	738048	3.22	815280	4.60	50
11	9.726426	3.34	9.798877	4.67	49	11	9.738241	3.22	9.815555	4.60	49
12	726626	3.34	799157	4.67	48	12	738434	3.22	815831	4.60	48
13	726827	3.34	799437	4.67	47	13	738627	3.22	816107	4.59	47
14	727027	3.34	799717	4.67	46	14	738820	3.21	816382	4.59	46
15	727228	3.33	799997	4.67	45	15	739013	3.21	816658	4.59	45
16	727428	3.33	800277	4.66	44	16	739206	3.21	816933	4.59	44
17	727628	3.33	800557	4.66	43	17	739398	3.21	817209	4.59	43
18	727828	3.33	800836	4.66	42	18	739590	3.20	817484	4.59	42
19	728027	3.33	801116	4.66	41	19	739783	3.20	817759	4.59	41
20	728227	3.33	801396	4.66	40	20	739975	3.20	818035	4.59	40
21	9.728427	3.32	9.801675	4.66	39	21	9.740167	3.20	9.818310	4.58	39
22	728626	3.32	801955	4.66	38	22	740359	3.20	818585	4.58	38
23	728825	3.32	802234	4.65	37	23	740550	3.19	818860	4.58	37
24	729024	3.32	802513	4.65	36	24	740742	3.19	819135	4.58	36
25	729223	3.32	802792	4.65	35	25	740934	3.19	819410	4.58	35
26	729422	3.31	803072	4.65	34	26	741125	3.19	819684	4.58	34
27	729621	3.31	803351	4.65	33	27	741316	3.19	819959	4.58	33
28	729820	3.31	803630	4.65	32	28	741508	3.18	820234	4.58	32
29	730018	3.30	803909	4.65	31	29	741699	3.18	820508	4.58	31
30	730217	3.30	804187	4.65	30	30	741889	3.18	820783	4.57	30
31	9.730415	3.30	9.804466	4.65	29	31	9.742080	3.18	9.821057	4.57	29
32	730613	3.30	804745	4.64	28	32	742271	3.18	821332	4.57	28
33	730811	3.30	805023	4.64	27	33	742462	3.17	821606	4.57	27
34	731009	3.29	805302	4.64	26	34	742652	3.17	821880	4.57	26
35	731206	3.29	805580	4.64	25	35	742842	3.17	822154	4.57	25
36	731404	3.29	805859	4.64	24	36	743033	3.17	822429	4.57	24
37	731602	3.29	806137	4.64	23	37	743223	3.17	822703	4.57	23
38	731799	3.29	806415	4.63	22	38	743413	3.16	822977	4.56	22
39	731996	3.28	806693	4.63	21	39	743602	3.16	823251	4.56	21
40	732193	3.28	806971	4.63	20	40	743792	3.16	823524	4.56	20
41	9.732390	3.28	9.807249	4.63	19	41	9.743982	3.16	9.823798	4.56	19
42	732587	3.28	807527	4.63	18	42	744171	3.16	824072	4.56	18
43	732784	3.28	807805	4.63	17	43	744361	3.15	824345	4.56	17
44	732980	3.27	808083	4.63	16	44	744550	3.15	824619	4.56	16
45	733177	3.27	808361	4.63	15	45	744739	3.15	824893	4.56	15
46	733373	3.27	808638	4.63	14	46	744928	3.15	825166	4.56	14
47	733569	3.27	808916	4.62	13	47	745117	3.15	825439	4.55	13
48	733765	3.27	809193	4.62	12	48	745306	3.14	825713	4.55	12
49	733961	3.26	809471	4.62	11	49	745494	3.14	825986	4.55	11
50	734157	3.26	809748	4.62	10	50	745683	3.14	826259	4.55	10
51	9.734353	3.26	9.810025	4.62	9	51	9.745871	3.14	9.826532	4.55	9
52	734549	3.26	810302	4.62	8	52	746060	3.14	826805	4.55	8
53	734744	3.25	810580	4.62	7	53	746248	3.13	827078	4.55	7
54	734939	3.25	810857	4.62	6	54	746436	3.13	827351	4.55	6
55	735135	3.25	811134	4.60	5	55	746624	3.13	827624	4.55	5
56	735330	3.25	811410	4.61	4	56	746812	3.13	827897	4.55	4
57	735525	3.25	811687	4.61	3	57	746999	3.13	828170	4.54	3
58	735719	3.25	811964	4.61	2	58	747187	3.13	828442	4.54	2
59	735914	3.24	812241	4.61	1	59	747374	3.12	828715	4.54	1
60	736109		812517		0	60	747562		828987		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.747562	3.12	9.828987	4.54	60	0	9.758591	3.01	9.845227	4.48	60
1	747749	3.12	829260	4.54	59	1	758772	3.00	845496	4.48	59
2	747936	3.12	829532	4.54	58	2	758952	3.00	845764	4.48	58
3	748123	3.12	829805	4.54	57	3	759132	3.00	846033	4.48	57
4	748310	3.11	830077	4.54	56	4	759312	3.00	846302	4.48	56
5	748497	3.11	830349	4.53	55	5	759492	3.00	846570	4.48	55
6	748683	3.11	830621	4.53	54	6	759672	3.00	846839	4.47	54
7	748870	3.11	830893	4.53	53	7	759852	2.99	847108	4.47	53
8	749056	3.11	831165	4.53	52	8	760031	2.99	847376	4.47	52
9	749243	3.10	831437	4.53	51	9	760211	2.99	847644	4.47	51
10	749429	3.10	831709	4.53	50	10	760390	2.99	847913	4.47	50
11	9.749615	3.10	9.831981	4.53	49	11	9.760569	2.98	9.848181	4.47	49
12	749801	3.10	832253	4.53	48	12	760748	2.98	848449	4.47	48
13	749987	3.09	832525	4.53	47	13	760927	2.98	848717	4.47	47
14	750172	3.09	832796	4.53	46	14	761106	2.98	848986	4.47	46
15	750358	3.09	833068	4.52	45	15	761285	2.98	849254	4.47	45
16	750543	3.09	833339	4.52	44	16	761464	2.98	849522	4.47	44
17	750729	3.09	833611	4.52	43	17	761642	2.97	849790	4.47	43
18	750914	3.08	833882	4.52	42	18	761821	2.97	850057	4.46	42
19	751099	3.08	834154	4.52	41	19	761999	2.97	850325	4.46	41
20	751284	3.08	834425	4.52	40	20	762177	2.97	850593	4.46	40
21	9.751469	3.08	9.834696	4.52	39	21	9.762356	2.97	9.850861	4.46	39
22	751654	3.08	834967	4.52	38	22	762534	2.97	851129	4.46	38
23	751839	3.08	835238	4.52	37	23	762712	2.96	851396	4.46	37
24	752023	3.07	835509	4.52	36	24	762889	2.96	851664	4.46	36
25	752208	3.07	835780	4.52	35	25	763067	2.96	851931	4.46	35
26	752392	3.07	836051	4.51	34	26	763245	2.96	852199	4.46	34
27	752576	3.07	836322	4.51	33	27	763422	2.96	852466	4.46	33
28	752760	3.07	836593	4.51	32	28	763600	2.95	852733	4.46	32
29	752944	3.07	836864	4.51	31	29	763777	2.95	853001	4.45	31
30	753128	3.06	837134	4.51	30	30	763954	2.95	853268	4.45	30
31	9.753312	3.06	9.837405	4.51	29	31	9.764131	2.95	9.853535	4.45	29
32	753495	3.06	837675	4.51	28	32	764308	2.95	853802	4.45	28
33	753679	3.06	837946	4.51	27	33	764485	2.95	854069	4.45	27
34	753862	3.05	838216	4.51	26	34	764662	2.94	854336	4.45	26
35	754046	3.05	838487	4.50	25	35	764838	2.94	854603	4.45	25
36	754229	3.05	838757	4.50	24	36	765015	2.94	854870	4.45	24
37	754412	3.05	839027	4.50	23	37	765191	2.94	855137	4.45	23
38	754595	3.05	839297	4.50	22	38	765367	2.94	855404	4.45	22
39	754778	3.04	839568	4.50	21	39	765544	2.93	855671	4.45	21
40	754960	3.04	839838	4.50	20	40	765720	2.93	855938	4.44	20
41	9.755143	3.04	9.840108	4.50	19	41	9.765896	2.93	9.856204	4.44	19
42	755326	3.04	840378	4.50	18	42	766072	2.93	856471	4.44	18
43	755508	3.04	840648	4.50	17	43	766247	2.93	856737	4.44	17
44	755690	3.04	840917	4.50	16	44	766423	2.93	857004	4.44	16
45	755872	3.03	841187	4.49	15	45	766598	2.92	857270	4.44	15
46	756054	3.03	841457	4.49	14	46	766774	2.92	857537	4.44	14
47	756236	3.03	841727	4.49	13	47	766949	2.92	857803	4.44	13
48	756418	3.03	841996	4.49	12	48	767124	2.92	858069	4.44	12
49	756600	3.03	842266	4.49	11	49	767300	2.92	858336	4.44	11
50	756782	3.02	842535	4.49	10	50	767475	2.91	858602	4.43	10
51	9.756963	3.02	9.842805	4.49	9	51	9.767649	2.91	9.858868	4.43	9
52	757144	3.02	843074	4.49	8	52	767824	2.91	859134	4.43	8
53	757326	3.02	843343	4.49	7	53	767999	2.91	859400	4.43	7
54	757507	3.02	843612	4.49	6	54	768173	2.91	859666	4.43	6
55	757688	3.02	843882	4.48	5	55	768348	2.90	859932	4.43	5
56	757869	3.01	844151	4.48	4	56	768522	2.90	860198	4.43	4
57	758050	3.01	844420	4.48	3	57	768697	2.90	860464	4.43	3
58	758230	3.01	844689	4.48	2	58	768871	2.90	860730	4.43	2
59	758411	3.01	844958	4.48	1	59	769045	2.90	860995	4.43	1
60	758591		845227		0	60	769219		861261		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.769219	2.90	9.861261	4.43	60	0	9.779463	2.79	9.877114	4.38	60
1	769393	2.89	861527	4.43	59	1	779631	2.79	877377	4.38	59
2	769566	2.89	861792	4.43	58	2	779798	2.79	877640	4.38	58
3	769740	2.89	862058	4.42	57	3	779966	2.79	877903	4.38	57
4	769913	2.89	862323	4.42	56	4	780133	2.79	878165	4.38	56
5	770087	2.89	862589	4.42	55	5	780300	2.78	878428	4.38	55
6	770260	2.88	862854	4.42	54	6	780467	2.78	878691	4.38	54
7	770433	2.88	863119	4.42	53	7	780634	2.78	878953	4.38	53
8	770606	2.88	863385	4.42	52	8	780801	2.78	879216	4.37	52
9	770779	2.88	863650	4.42	51	9	780968	2.78	879478	4.37	51
10	770952	2.88	863915	4.42	50	10	781134	2.78	879741	4.37	50
11	9.771125	2.88	9.864180	4.42	49	11	9.781301	2.78	9.880003	4.37	49
12	771298	2.87	864445	4.42	48	12	781468	2.77	880265	4.37	48
13	771470	2.87	864710	4.42	47	13	781634	2.77	880528	4.37	47
14	771643	2.87	864975	4.42	46	14	781800	2.77	880790	4.37	46
15	771815	2.87	865240	4.41	45	15	781966	2.77	881052	4.37	45
16	771987	2.87	865505	4.41	44	16	782132	2.77	881314	4.37	44
17	772159	2.87	865770	4.41	43	17	782298	2.77	881577	4.37	43
18	772331	2.87	866035	4.41	42	18	782464	2.76	881839	4.37	42
19	772503	2.86	866300	4.41	41	19	782630	2.76	882101	4.37	41
20	772675	2.86	866564	4.41	40	20	782796	2.76	882363	4.37	40
21	9.772847	2.86	9.866829	4.41	39	21	9.782961	2.76	9.882625	4.36	39
22	773018	2.86	867094	4.41	38	22	783127	2.76	882887	4.36	38
23	773190	2.86	867358	4.41	37	23	783292	2.75	883148	4.36	37
24	773361	2.85	867623	4.41	36	24	783458	2.75	883410	4.36	36
25	773533	2.85	867887	4.41	35	25	783623	2.75	883672	4.36	35
26	773704	2.85	868152	4.40	34	26	783788	2.75	883934	4.36	34
27	773875	2.85	868416	4.40	33	27	783953	2.75	884196	4.36	33
28	774046	2.85	868680	4.40	32	28	784118	2.75	884457	4.36	32
29	774217	2.85	868945	4.40	31	29	784282	2.75	884719	4.36	31
30	774388	2.84	869209	4.40	30	30	784447	2.74	884980	4.36	30
31	9.774558	2.84	9.869473	4.40	29	31	9.784612	2.74	9.885242	4.36	29
32	774729	2.84	869737	4.40	28	32	784776	2.74	885504	4.36	28
33	774899	2.84	870001	4.40	27	33	784941	2.74	885765	4.36	27
34	775070	2.84	870265	4.40	26	34	785105	2.74	886026	4.36	26
35	775240	2.84	870529	4.40	25	35	785269	2.73	886288	4.36	25
36	775410	2.83	870793	4.40	24	36	785433	2.73	886549	4.35	24
37	775580	2.83	871057	4.40	23	37	785597	2.73	886811	4.35	23
38	775750	2.83	871321	4.40	22	38	785761	2.73	887072	4.35	22
39	775920	2.83	871585	4.40	21	39	785925	2.73	887333	4.35	21
40	776090	2.83	871849	4.39	20	40	786089	2.73	887594	4.35	20
41	9.776259	2.83	9.872112	4.39	19	41	9.786252	2.73	9.887855	4.35	19
42	776429	2.82	872376	4.39	18	42	786416	2.72	888116	4.35	18
43	776598	2.82	872640	4.39	17	43	786579	2.72	888378	4.35	17
44	776768	2.82	872903	4.39	16	44	786742	2.72	888639	4.35	16
45	776937	2.82	873167	4.39	15	45	786906	2.72	888900	4.35	15
46	777106	2.82	873430	4.39	14	46	787069	2.72	889161	4.35	14
47	777275	2.82	873694	4.39	13	47	787232	2.72	889421	4.35	13
48	777444	2.81	873957	4.39	12	48	787395	2.71	889682	4.35	12
49	777613	2.81	874220	4.39	11	49	787557	2.71	889943	4.35	11
50	777781	2.81	874484	4.39	10	50	787720	2.71	890204	4.35	10
51	9.777950	2.81	9.874747	4.39	9	51	9.787883	2.71	9.890465	4.34	9
52	778119	2.81	875010	4.39	8	52	788045	2.71	890725	4.34	8
53	778287	2.80	875273	4.38	7	53	788208	2.71	890986	4.34	7
54	778455	2.80	875537	4.38	6	54	788370	2.70	891247	4.34	6
55	778624	2.80	875800	4.38	5	55	788532	2.70	891507	4.34	5
56	778792	2.80	876063	4.38	4	56	788694	2.70	891768	4.34	4
57	778960	2.80	876326	4.38	3	57	788856	2.70	892028	4.34	3
58	779128	2.80	876589	4.38	2	58	789018	2.70	892289	4.34	2
59	779295	2.80	876852	4.38	1	59	789180	2.70	892549	4.34	1
60	779463		877114		0	60	789342		892810		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.789342	2.69	9.892810	4.34	60	0	9.798872	2.60	9.908369	4.30	60
1	789504	2.69	893070	4.34	59	1	799028	2.60	908628	4.30	59
2	789665	2.69	893331	4.34	58	2	799184	2.60	908886	4.30	58
3	789827	2.69	893591	4.34	57	3	799339	2.60	909144	4.30	57
4	789988	2.69	893851	4.34	56	4	799495	2.59	909402	4.30	56
5	790149	2.69	894111	4.34	55	5	799651	2.59	909660	4.30	55
6	790310	2.68	894372	4.34	54	6	799806	2.59	909918	4.30	54
7	790471	2.68	894632	4.33	53	7	799962	2.59	910177	4.30	53
8	790632	2.68	894892	4.33	52	8	800117	2.59	910435	4.30	52
9	790793	2.68	895152	4.33	51	9	800272	2.58	910693	4.30	51
10	790954	2.68	895412	4.33	50	10	800427	2.58	910951	4.30	50
11	9.791115	2.68	9.895672	4.33	49	11	9.800582	2.58	9.911209	4.30	49
12	791275	2.68	895932	4.33	48	12	800737	2.58	911467	4.30	48
13	791436	2.67	896192	4.33	47	13	800892	2.58	911725	4.30	47
14	791596	2.67	896452	4.33	46	14	801047	2.58	911982	4.30	46
15	791757	2.67	896712	4.33	45	15	801201	2.58	912240	4.30	45
16	791917	2.67	896971	4.33	44	16	801356	2.58	912498	4.30	44
17	792077	2.67	897231	4.33	43	17	801511	2.57	912756	4.30	43
18	792237	2.67	897491	4.33	42	18	801665	2.57	913014	4.29	42
19	792397	2.66	897751	4.33	41	19	801819	2.57	913271	4.29	41
20	792557	2.66	898010	4.33	40	20	801973	2.57	913529	4.29	40
21	9.792716	2.66	9.898270	4.33	39	21	9.802128	2.57	9.913787	4.29	39
22	792875	2.66	898530	4.33	38	22	802282	2.57	914044	4.29	38
23	793035	2.66	898789	4.33	37	23	802436	2.56	914302	4.29	37
24	793195	2.65	899049	4.32	36	24	802589	2.56	914560	4.29	36
25	793354	2.65	899308	4.32	35	25	802743	2.56	914817	4.29	35
26	793514	2.65	899568	4.32	34	26	802897	2.56	915075	4.29	34
27	793673	2.65	899827	4.32	33	27	803050	2.56	915332	4.29	33
28	793832	2.65	900087	4.32	32	28	803204	2.56	915590	4.29	32
29	793991	2.65	900346	4.32	31	29	803357	2.55	915847	4.29	31
30	794150	2.64	900605	4.32	30	30	803511	2.55	916104	4.29	30
31	9.794308	2.64	9.900864	4.32	29	31	9.803664	2.55	9.916362	4.29	29
32	794467	2.64	901124	4.32	28	32	803817	2.55	916619	4.29	28
33	794626	2.64	901383	4.32	27	33	803970	2.55	916877	4.29	27
34	794784	2.64	901642	4.32	26	34	804123	2.55	917134	4.29	26
35	794942	2.64	901901	4.32	25	35	804276	2.54	917391	4.29	25
36	795101	2.64	902160	4.32	24	36	804428	2.54	917648	4.29	24
37	795259	2.63	902420	4.32	23	37	804581	2.54	917906	4.29	23
38	795417	2.63	902679	4.32	22	38	804734	2.54	918163	4.28	22
39	795575	2.63	902938	4.32	21	39	804886	2.54	918420	4.28	21
40	795733	2.63	903197	4.31	20	40	805039	2.54	918677	4.28	20
41	9.795891	2.63	9.903456	4.31	19	41	9.805191	2.54	9.918934	4.28	19
42	796049	2.63	903714	4.31	18	42	805343	2.53	919191	4.28	18
43	796206	2.63	903973	4.31	17	43	805495	2.53	919448	4.28	17
44	796364	2.62	904232	4.31	16	44	805647	2.53	919705	4.28	16
45	796521	2.62	904491	4.31	15	45	805799	2.53	919962	4.28	15
46	796679	2.62	904750	4.31	14	46	805951	2.53	920219	4.28	14
47	796836	2.62	905008	4.31	13	47	806103	2.53	920476	4.28	13
48	796993	2.62	905267	4.31	12	48	806254	2.53	920733	4.28	12
49	797150	2.62	905526	4.31	11	49	806406	2.52	920990	4.28	11
50	797307	2.62	905785	4.31	10	50	806557	2.52	921247	4.28	10
51	9.797464	2.61	9.906043	4.31	9	51	9.806709	2.52	9.921503	4.28	9
52	797621	2.61	906302	4.31	8	52	806860	2.52	921760	4.28	8
53	797777	2.61	906560	4.31	7	53	807011	2.52	922017	4.28	7
54	797934	2.61	906819	4.31	6	54	807163	2.52	922274	4.28	6
55	798091	2.61	907077	4.31	5	55	807314	2.52	922530	4.28	5
56	798247	2.61	907336	4.31	4	56	807465	2.51	922787	4.28	4
57	798403	2.60	907594	4.31	3	57	807615	2.51	923044	4.28	3
58	798560	2.60	907853	4.31	2	58	807766	2.51	923300	4.28	2
59	798716	2.60	908111	4.30	1	59	807917	2.51	923557	4.27	1
60	798872		908369		0	60	808067		923814		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.808067	2.51	9.923814	4.27	60	0	9.816943	2.42	9.939163	4.25	60
1	808218	2.51	924070	4.27	59	1	817088	2.42	939418	4.25	59
2	808368	2.51	924327	4.27	58	2	817233	2.42	939673	4.25	58
3	808519	2.50	924583	4.27	57	3	817379	2.42	939928	4.25	57
4	808669	2.50	924840	4.27	56	4	817524	2.41	940183	4.25	56
5	808819	2.50	925096	4.27	55	5	817668	2.41	940439	4.25	55
6	808969	2.50	925352	4.27	54	6	817813	2.41	940694	4.25	54
7	809119	2.50	925609	4.27	53	7	817958	2.41	940949	4.25	53
8	809269	2.50	925865	4.27	52	8	818103	2.41	941204	4.25	52
9	809419	2.50	926122	4.27	51	9	818247	2.41	941459	4.25	51
10	809569	2.49	926378	4.27	50	10	818392	2.41	941713	4.25	50
11	9.809718	2.49	9.926634	4.27	49	11	9.818536	2.41	9.941968	4.25	49
12	809868	2.49	926890	4.27	48	12	818681	2.40	942223	4.25	48
13	810017	2.49	927147	4.27	47	13	818825	2.40	942478	4.25	47
14	810167	2.49	927403	4.27	46	14	818969	2.40	942733	4.25	46
15	810316	2.48	927659	4.27	45	15	819113	2.40	942988	4.25	45
16	810465	2.48	927915	4.27	44	16	819257	2.40	943243	4.25	44
17	810614	2.48	928171	4.27	43	17	819401	2.40	943498	4.25	43
18	810763	2.48	928427	4.27	42	18	819545	2.40	943752	4.25	42
19	810912	2.48	928684	4.27	41	19	819689	2.39	944007	4.25	41
20	811061	2.48	928940	4.27	40	20	819832	2.39	944262	4.25	40
21	9.811210	2.48	9.929196	4.27	39	21	9.819976	2.39	9.944517	4.25	39
22	811358	2.48	929452	4.27	38	22	820120	2.39	944771	4.25	38
23	811507	2.47	929708	4.27	37	23	820263	2.39	945026	4.25	37
24	811655	2.47	929964	4.27	36	24	820406	2.39	945281	4.24	36
25	811804	2.47	930220	4.26	35	25	820550	2.38	945535	4.24	35
26	811952	2.47	930475	4.26	34	26	820693	2.38	945790	4.24	34
27	812100	2.47	930731	4.26	33	27	820836	2.38	946045	4.24	33
28	812248	2.47	930987	4.26	32	28	820979	2.38	946299	4.24	32
29	812396	2.46	931243	4.26	31	29	821122	2.38	946554	4.24	31
30	812544	2.46	931499	4.26	30	30	821265	2.38	946808	4.24	30
31	9.812692	2.46	9.931755	4.26	29	31	9.821407	2.38	9.947063	4.24	29
32	812840	2.46	932010	4.26	28	32	821550	2.38	947318	4.24	28
33	812988	2.46	932266	4.26	27	33	821693	2.37	947572	4.24	27
34	813135	2.46	932522	4.26	26	34	821835	2.37	947827	4.24	26
35	813283	2.46	932778	4.26	25	35	821977	2.37	948081	4.24	25
36	813430	2.46	933033	4.26	24	36	822120	2.37	948335	4.24	24
37	813578	2.45	933289	4.26	23	37	822262	2.37	948590	4.24	23
38	813725	2.45	933545	4.26	22	38	822404	2.37	948844	4.24	22
39	813872	2.45	933800	4.26	21	39	822546	2.37	949099	4.24	21
40	814019	2.45	934056	4.26	20	40	822688	2.37	949353	4.24	20
41	9.814166	2.45	9.934311	4.26	19	41	9.822830	2.36	9.949608	4.24	19
42	814313	2.45	934567	4.26	18	42	822972	2.36	949862	4.24	18
43	814460	2.45	934822	4.26	17	43	823114	2.36	950116	4.24	17
44	814607	2.44	935078	4.26	16	44	823255	2.36	950371	4.24	16
45	814753	2.44	935333	4.26	15	45	823397	2.36	950625	4.24	15
46	814900	2.44	935589	4.26	14	46	823539	2.36	950879	4.24	14
47	815046	2.44	935844	4.26	13	47	823680	2.35	951133	4.24	13
48	815193	2.44	936100	4.26	12	48	823821	2.35	951388	4.24	12
49	815339	2.44	936355	4.26	11	49	823963	2.35	951642	4.24	11
50	815485	2.43	936611	4.26	10	50	824104	2.35	951896	4.24	10
51	9.815632	2.43	9.936866	4.25	9	51	9.824245	2.35	9.952150	4.24	9
52	815778	2.43	937121	4.25	8	52	824386	2.35	952405	4.24	8
53	815924	2.43	937377	4.25	7	53	824527	2.35	952659	4.24	7
54	816069	2.43	937632	4.25	6	54	824668	2.34	952913	4.24	6
55	816215	2.43	937887	4.25	5	55	824808	2.34	953167	4.23	5
56	816361	2.43	938142	4.25	4	56	824949	2.34	953421	4.23	4
57	816507	2.42	938398	4.25	3	57	825090	2.34	953675	4.23	3
58	816652	2.42	938653	4.25	2	58	825230	2.34	953929	4.23	2
59	816798	2.42	938908	4.25	1	59	825371	2.34	954183	4.23	1
60	816943		939163		0	60	825511		954437		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.825511	2.34	9.954437	4.23	60	0	9.833783	2.26	9.969656	4.22	60
1	825651	2.33	954691	4.23	59	1	833919	2.25	969909	4.22	59
2	825791	2.33	954946	4.23	58	2	834054	2.25	970162	4.22	58
3	825931	2.33	955200	4.23	57	3	834189	2.25	970416	4.22	57
4	826071	2.33	955454	4.23	56	4	834325	2.25	970669	4.22	56
5	826211	2.33	955708	4.23	55	5	834460	2.25	970922	4.22	55
6	826351	2.33	955961	4.23	54	6	834595	2.25	971175	4.22	54
7	826491	2.33	956215	4.23	53	7	834730	2.25	971429	4.22	53
8	826631	2.33	956469	4.23	52	8	834865	2.25	971682	4.22	52
9	826770	2.33	956723	4.23	51	9	834999	2.25	971935	4.22	51
10	826910	2.32	956977	4.23	50	10	835134	2.24	972188	4.22	50
11	9.827049	2.32	9.957231	4.23	49	11	9.835269	2.24	9.972441	4.22	49
12	827189	2.32	957485	4.23	48	12	835403	2.24	972695	4.22	48
13	827328	2.32	957739	4.23	47	13	835538	2.24	972948	4.22	47
14	827467	2.32	957993	4.23	46	14	835672	2.24	973201	4.22	46
15	827606	2.32	958247	4.23	45	15	835807	2.24	973454	4.22	45
16	827745	2.32	958500	4.23	44	16	835941	2.24	973707	4.22	44
17	827884	2.32	958754	4.23	43	17	836075	2.23	973960	4.22	43
18	828023	2.31	959008	4.23	42	18	836209	2.23	974213	4.22	42
19	828162	2.31	959262	4.23	41	19	836343	2.23	974466	4.22	41
20	828301	2.31	959516	4.23	40	20	836477	2.23	974720	4.22	40
21	9.828439	2.31	9.959769	4.23	39	21	9.836611	2.23	9.974973	4.22	39
22	828578	2.31	960023	4.23	38	22	836745	2.23	975226	4.22	38
23	828716	2.31	960277	4.23	37	23	836878	2.23	975479	4.22	37
24	828855	2.30	960530	4.23	36	24	837012	2.22	975732	4.22	36
25	828993	2.30	960784	4.23	35	25	837146	2.22	975985	4.22	35
26	829131	2.30	961038	4.23	34	26	837279	2.22	976238	4.22	34
27	829269	2.30	961292	4.23	33	27	837412	2.22	976491	4.22	33
28	829407	2.30	961545	4.23	32	28	837546	2.22	976744	4.22	32
29	829545	2.30	961799	4.23	31	29	837679	2.22	976997	4.22	31
30	829683	2.30	962052	4.23	30	30	837812	2.22	977250	4.22	30
31	9.829821	2.30	9.962306	4.23	29	31	9.837945	2.22	9.977503	4.22	29
32	829959	2.29	962560	4.23	28	32	838078	2.22	977756	4.22	28
33	830097	2.29	962813	4.23	27	33	838211	2.21	978009	4.22	27
34	830234	2.29	963067	4.23	26	34	838344	2.21	978262	4.22	26
35	830372	2.29	963320	4.23	25	35	838477	2.21	978515	4.22	25
36	830509	2.29	963574	4.23	24	36	838610	2.21	978768	4.22	24
37	830646	2.29	963828	4.23	23	37	838742	2.21	979021	4.22	23
38	830784	2.29	964081	4.23	22	38	838875	2.21	979274	4.22	22
39	830921	2.28	964335	4.23	21	39	839007	2.21	979527	4.22	21
40	831058	2.28	964588	4.23	20	40	839140	2.20	979780	4.22	20
41	9.831195	2.28	9.964842	4.22	19	41	9.839272	2.20	9.980033	4.22	19
42	831332	2.28	965095	4.22	18	42	839404	2.20	980286	4.22	18
43	831469	2.28	965349	4.22	17	43	839536	2.20	980538	4.22	17
44	831606	2.28	965602	4.22	16	44	839668	2.20	980791	4.22	16
45	831742	2.28	965855	4.22	15	45	839800	2.20	981044	4.21	15
46	831879	2.28	966109	4.22	14	46	839932	2.20	981297	4.21	14
47	832015	2.28	966362	4.22	13	47	840064	2.20	981550	4.21	13
48	832152	2.27	966616	4.22	12	48	840196	2.19	981803	4.21	12
49	832288	2.27	966869	4.22	11	49	840328	2.19	982056	4.21	11
50	832425	2.27	967123	4.22	10	50	840459	2.19	982309	4.21	10
51	9.832561	2.27	9.967376	4.22	9	51	9.840591	2.19	9.982562	4.21	9
52	832697	2.27	967629	4.22	8	52	840722	2.19	982814	4.21	8
53	832833	2.27	967883	4.22	7	53	840854	2.19	983067	4.21	7
54	832969	2.27	968136	4.22	6	54	840985	2.19	983320	4.21	6
55	833105	2.26	968389	4.22	5	55	841116	2.18	983573	4.21	5
56	833241	2.26	968643	4.22	4	56	841247	2.18	983826	4.21	4
57	833377	2.26	968896	4.22	3	57	841378	2.18	984079	4.21	3
58	833512	2.26	969149	4.22	2	58	841509	2.18	984332	4.21	2
59	833648	2.26	969403	4.22	1	59	841640	2.18	984584	4.21	1
60	833783		969656		0	60	841771		984837		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.841771	2.18	9.984837	4.21	60	0	9.849485	2.10	10.000000	4.21	60
1	841902	2.18	985090	4.21	59	1	849611	2.10	000253	4.21	59
2	842033	2.18	985343	4.21	58	2	849738	2.10	000505	4.21	58
3	842163	2.18	985596	4.21	57	3	849864	2.10	000758	4.21	57
4	842294	2.17	985848	4.21	56	4	849990	2.10	001011	4.21	56
5	842424	2.17	986101	4.21	55	5	850116	2.10	001263	4.21	55
6	842555	2.17	986354	4.21	54	6	850242	2.10	001516	4.21	54
7	842685	2.17	986607	4.21	53	7	850368	2.10	001769	4.21	53
8	842815	2.17	986860	4.21	52	8	850493	2.10	002021	4.21	52
9	842946	2.17	987112	4.21	51	9	850619	2.09	002274	4.21	51
10	843076	2.17	987365	4.21	50	10	850745	2.09	002527	4.21	50
11	9.843206	2.17	9.987618	4.21	49	11	9.850870	2.09	10.002779	4.21	49
12	843336	2.16	987871	4.21	48	12	850996	2.09	003032	4.21	48
13	843466	2.16	988123	4.21	47	13	851121	2.09	003285	4.21	47
14	843595	2.16	988376	4.21	46	14	851246	2.09	003537	4.21	46
15	843725	2.16	988629	4.21	45	15	851372	2.09	003790	4.21	45
16	843855	2.16	988882	4.21	44	16	851497	2.08	004043	4.21	44
17	843984	2.16	989134	4.21	43	17	851622	2.08	004295	4.21	43
18	844114	2.15	989387	4.21	42	18	851747	2.08	004548	4.21	42
19	844243	2.15	989640	4.21	41	19	851872	2.08	004801	4.21	41
20	844372	2.15	989893	4.21	40	20	851997	2.08	005053	4.21	40
21	9.844502	2.15	9.990145	4.21	39	21	9.852122	2.08	10.005306	4.21	39
22	844631	2.15	990398	4.21	38	22	852247	2.08	005559	4.21	38
23	844760	2.15	990651	4.21	37	23	852371	2.08	005811	4.21	37
24	844889	2.15	990903	4.21	36	24	852496	2.07	006064	4.21	36
25	845018	2.15	991156	4.21	35	25	852620	2.07	006317	4.21	35
26	845147	2.15	991409	4.21	34	26	852745	2.07	006569	4.21	34
27	845276	2.15	991662	4.21	33	27	852869	2.07	006822	4.21	33
28	845405	2.14	991914	4.21	32	28	852994	2.07	007075	4.21	32
29	845533	2.14	992167	4.21	31	29	853118	2.07	007328	4.21	31
30	845662	2.14	992420	4.21	30	30	853242	2.07	007580	4.21	30
31	9.845790	2.14	9.992672	4.21	29	31	9.853366	2.07	10.007833	4.21	29
32	845919	2.14	992925	4.21	28	32	853490	2.07	008086	4.21	28
33	846047	2.14	993178	4.21	27	33	853614	2.07	008338	4.21	27
34	846175	2.14	993431	4.21	26	34	853738	2.06	008591	4.21	26
35	846304	2.14	993683	4.21	25	35	853862	2.06	008844	4.21	25
36	846432	2.13	993936	4.21	24	36	853986	2.06	009097	4.21	24
37	846560	2.13	994189	4.21	23	37	854109	2.06	009349	4.21	23
38	846688	2.13	994441	4.21	22	38	854233	2.06	009602	4.21	22
39	846816	2.13	994694	4.21	21	39	854356	2.06	009855	4.21	21
40	846944	2.13	994947	4.21	20	40	854480	2.06	010107	4.21	20
41	9.847071	2.13	9.995199	4.21	19	41	9.854603	2.05	10.010360	4.21	19
42	847199	2.13	995452	4.21	18	42	854727	2.05	010613	4.21	18
43	847327	2.13	995705	4.21	17	43	854850	2.05	010866	4.21	17
44	847454	2.13	995957	4.21	16	44	854973	2.05	011118	4.21	16
45	847582	2.12	996210	4.21	15	45	855096	2.05	011371	4.21	15
46	847709	2.12	996463	4.21	14	46	855219	2.05	011624	4.21	14
47	847836	2.12	996715	4.21	13	47	855342	2.05	011877	4.21	13
48	847964	2.12	996968	4.21	12	48	855465	2.05	012129	4.21	12
49	848091	2.12	997221	4.21	11	49	855588	2.05	012382	4.21	11
50	848218	2.12	997473	4.21	10	50	855711	2.04	012635	4.21	10
51	9.848345	2.12	9.997726	4.21	9	51	9.855833	2.04	10.012888	4.21	9
52	848472	2.12	997979	4.21	8	52	855956	2.04	013140	4.21	8
53	848599	2.11	998231	4.21	7	53	856078	2.04	013393	4.21	7
54	848726	2.11	998484	4.21	6	54	856201	2.04	013646	4.21	6
55	848852	2.11	998737	4.21	5	55	856323	2.04	013899	4.21	5
56	848979	2.11	998989	4.21	4	56	856446	2.04	014152	4.21	4
57	849106	2.11	999242	4.21	3	57	856568	2.04	014404	4.21	3
58	849232	2.11	999495	4.21	2	58	856690	2.03	014657	4.21	2
59	849359	2.10	999747	4.21	1	59	856812	2.03	014910	4.21	1
60	849485		10.000000		0	60	856934		015163		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.856934	2.03	10.015163	4.21	60	0	9.864127	1.96	10.030344	4.22	60
1	857056	2.03	015416	4.21	59	1	864245	1.96	030597	4.22	59
2	857178	2.03	015668	4.21	58	2	864363	1.96	030851	4.22	58
3	857300	2.03	015921	4.21	57	3	864481	1.96	031104	4.22	57
4	857422	2.03	016174	4.21	56	4	864598	1.96	031357	4.22	56
5	857543	2.03	016427	4.21	55	5	864716	1.96	031611	4.22	55
6	857665	2.02	016680	4.21	54	6	864833	1.96	031864	4.22	54
7	857786	2.02	016933	4.21	53	7	864950	1.95	032117	4.22	53
8	857908	2.02	017186	4.21	52	8	865068	1.95	032371	4.22	52
9	858029	2.02	017438	4.21	51	9	865185	1.95	032624	4.22	51
10	858151	2.02	017691	4.21	50	10	865302	1.95	032877	4.22	50
11	9.858272	2.02	10.017944	4.21	49	11	9.865419	1.95	10.033131	4.22	49
12	858393	2.02	018197	4.21	48	12	865536	1.95	033384	4.22	48
13	858514	2.02	018450	4.21	47	13	865653	1.95	033638	4.22	47
14	858635	2.02	018703	4.21	46	14	865770	1.95	033891	4.22	46
15	858756	2.02	018956	4.22	45	15	865887	1.95	034145	4.22	45
16	858877	2.01	019209	4.22	44	16	866004	1.94	034398	4.22	44
17	858998	2.01	019462	4.22	43	17	866120	1.94	034651	4.22	43
18	859119	2.01	019714	4.22	42	18	866237	1.94	034905	4.22	42
19	859239	2.01	019967	4.22	41	19	866353	1.94	035158	4.23	41
20	859360	2.01	020220	4.22	40	20	866470	1.94	035412	4.23	40
21	9.859480	2.01	10.020473	4.22	39	21	9.866586	1.94	10.035665	4.23	39
22	859601	2.01	020726	4.22	38	22	866703	1.94	035919	4.23	38
23	859721	2.01	020979	4.22	37	23	866819	1.94	036172	4.23	37
24	859842	2.00	021232	4.22	36	24	866935	1.93	036426	4.23	36
25	859962	2.00	021485	4.22	35	25	867051	1.93	036680	4.23	35
26	860082	2.00	021738	4.22	34	26	867167	1.93	036933	4.23	34
27	860202	2.00	021991	4.22	33	27	867283	1.93	037187	4.23	33
28	860322	2.00	022244	4.22	32	28	867399	1.93	037440	4.23	32
29	860442	2.00	022497	4.22	31	29	867515	1.93	037694	4.23	31
30	860562	2.00	022750	4.22	30	30	867631	1.93	037948	4.23	30
31	9.860682	2.00	10.023003	4.22	29	31	9.867747	1.93	10.038201	4.23	29
32	860802	1.99	023256	4.22	28	32	867862	1.93	038455	4.23	28
33	860922	1.99	023509	4.22	27	33	867978	1.92	038708	4.23	27
34	861041	1.99	023762	4.22	26	34	868093	1.92	038962	4.23	26
35	861161	1.99	024015	4.22	25	35	868209	1.92	039216	4.23	25
36	861280	1.99	024268	4.22	24	36	868324	1.92	039470	4.23	24
37	861400	1.99	024521	4.22	23	37	868440	1.92	039723	4.23	23
38	861519	1.99	024774	4.22	22	38	868555	1.92	039977	4.23	22
39	861638	1.99	025027	4.22	21	39	868670	1.92	040231	4.23	21
40	861758	1.98	025280	4.22	20	40	868785	1.92	040484	4.23	20
41	9.861877	1.98	10.025534	4.22	19	41	9.868900	1.92	10.040738	4.23	19
42	861996	1.98	025787	4.22	18	42	869015	1.92	040992	4.23	18
43	862115	1.98	026040	4.22	17	43	869130	1.91	041246	4.23	17
44	862234	1.98	026293	4.22	16	44	869245	1.91	041500	4.23	16
45	862353	1.98	026546	4.22	15	45	869360	1.91	041753	4.23	15
46	862471	1.98	026799	4.22	14	46	869474	1.91	042007	4.23	14
47	862590	1.98	027052	4.22	13	47	869589	1.91	042261	4.23	13
48	862709	1.98	027305	4.22	12	48	869704	1.91	042515	4.23	12
49	862827	1.98	027559	4.22	11	49	869818	1.91	042769	4.23	11
50	862946	1.97	027812	4.22	10	50	869933	1.91	043023	4.23	10
51	9.863064	1.97	10.028065	4.22	9	51	9.870047	1.90	10.043277	4.23	9
52	863183	1.97	028318	4.22	8	52	870161	1.90	043531	4.23	8
53	863301	1.97	028571	4.22	7	53	870276	1.90	043785	4.23	7
54	863419	1.97	028825	4.22	6	54	870390	1.90	044039	4.23	6
55	863538	1.97	029078	4.22	5	55	870504	1.90	044292	4.23	5
56	863656	1.97	029331	4.22	4	56	870618	1.90	044546	4.23	4
57	863774	1.97	029584	4.22	3	57	870732	1.90	044800	4.23	3
58	863892	1.97	029838	4.22	2	58	870846	1.90	045054	4.23	2
59	864010	1.96	030091	4.22	1	59	870960	1.90	045309	4.23	1
60	864127		030344		0	60	871073		045563		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.871073	1.90	10.045563	4.23	60	0	9.877780	1.83	10.060837	4.25	60
1	871187	1.89	045817	4.23	59	1	877890	1.83	061092	4.25	59
2	871301	1.89	046071	4.23	58	2	877999	1.83	061347	4.25	58
3	871414	1.89	046325	4.23	57	3	878109	1.83	061602	4.25	57
4	871528	1.89	046579	4.23	56	4	878219	1.82	061858	4.25	56
5	871641	1.89	046833	4.24	55	5	878328	1.82	062113	4.25	55
6	871755	1.89	047087	4.24	54	6	878438	1.82	062368	4.25	54
7	871868	1.89	047341	4.24	53	7	878547	1.82	062623	4.25	53
8	871981	1.89	047595	4.24	52	8	878656	1.82	062879	4.25	52
9	872095	1.88	047850	4.24	51	9	878766	1.82	063134	4.26	51
10	872208	1.88	048104	4.24	50	10	878875	1.82	063389	4.26	50
11	9.872321	1.88	10.048358	4.24	49	11	9.878984	1.82	10.063645	4.26	49
12	872434	1.88	048612	4.24	48	12	879093	1.82	063900	4.26	48
13	872547	1.88	048867	4.24	47	13	879202	1.82	064156	4.26	47
14	872659	1.88	049121	4.24	46	14	879311	1.81	064411	4.26	46
15	872772	1.88	049375	4.24	45	15	879420	1.81	064667	4.26	45
16	872885	1.88	049629	4.24	44	16	879529	1.81	064922	4.26	44
17	872998	1.88	049884	4.24	43	17	879637	1.81	065178	4.26	43
18	873110	1.88	050138	4.24	42	18	879746	1.81	065433	4.26	42
19	873223	1.87	050392	4.24	41	19	879855	1.81	065689	4.26	41
20	873335	1.87	050647	4.24	40	20	879963	1.81	065944	4.26	40
21	9.873448	1.87	10.050901	4.24	39	21	9.880072	1.81	10.066200	4.26	39
22	873560	1.87	051156	4.24	38	22	880180	1.81	066455	4.26	38
23	873672	1.87	051410	4.24	37	23	880289	1.80	066711	4.26	37
24	873784	1.87	051665	4.24	36	24	880397	1.80	066967	4.26	36
25	873896	1.87	051919	4.24	35	25	880505	1.80	067222	4.26	35
26	874009	1.87	052173	4.24	34	26	880613	1.80	067478	4.26	34
27	874121	1.87	052428	4.24	33	27	880722	1.80	067734	4.26	33
28	874232	1.87	052682	4.24	32	28	880830	1.80	067990	4.26	32
29	874344	1.86	052937	4.24	31	29	880938	1.80	068245	4.26	31
30	874456	1.86	053192	4.24	30	30	881046	1.80	068501	4.26	30
31	9.874568	1.86	10.053446	4.24	29	31	9.881153	1.80	10.068757	4.26	29
32	874680	1.86	053701	4.24	28	32	881261	1.80	069013	4.26	28
33	874791	1.86	053955	4.24	27	33	881369	1.79	069269	4.26	27
34	874903	1.86	054210	4.24	26	34	881477	1.79	069525	4.26	26
35	875014	1.86	054465	4.24	25	35	881584	1.79	069780	4.27	25
36	875126	1.85	054719	4.25	24	36	881692	1.79	070036	4.27	24
37	875237	1.85	054974	4.25	23	37	881799	1.79	070292	4.27	23
38	875348	1.85	055229	4.25	22	38	881907	1.79	070548	4.27	22
39	875459	1.85	055483	4.25	21	39	882014	1.79	070804	4.27	21
40	875571	1.85	055738	4.25	20	40	882121	1.79	071060	4.27	20
41	9.875682	1.85	10.055993	4.25	19	41	9.882229	1.79	10.071316	4.27	19
42	875793	1.85	056248	4.25	18	42	882336	1.78	071573	4.27	18
43	875904	1.85	056502	4.25	17	43	882443	1.78	071829	4.27	17
44	876014	1.85	056757	4.25	16	44	882550	1.78	072085	4.27	16
45	876125	1.85	057012	4.25	15	45	882657	1.78	072341	4.27	15
46	876236	1.85	057267	4.25	14	46	882764	1.78	072597	4.27	14
47	876347	1.84	057522	4.25	13	47	882871	1.78	072853	4.27	13
48	876457	1.84	057777	4.25	12	48	882977	1.78	073110	4.27	12
49	876568	1.84	058032	4.25	11	49	883084	1.78	073366	4.27	11
50	876678	1.84	058287	4.25	10	50	883191	1.78	073622	4.27	10
51	9.876789	1.84	10.058541	4.25	9	51	9.883297	1.77	10.073878	4.27	9
52	876899	1.84	058796	4.25	8	52	883404	1.77	074135	4.27	8
53	877010	1.84	059051	4.25	7	53	883510	1.77	074391	4.27	7
54	877120	1.84	059306	4.25	6	54	883617	1.77	074648	4.27	6
55	877230	1.83	059561	4.25	5	55	883723	1.77	074904	4.27	5
56	877340	1.83	059817	4.25	4	56	883829	1.77	075160	4.27	4
57	877450	1.83	060072	4.25	3	57	883936	1.77	075417	4.27	3
58	877560	1.83	060327	4.25	2	58	884042	1.77	075673	4.27	2
59	877670	1.83	060582	4.25	1	59	884148	1.77	075930	4.27	1
60	877780		060837		0	60	884254		076186		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.884254	1.77	10.076186	4.27	60	0	9.890503	1.70	10.091631	4.30	60
1	884360	1.76	076443	4.28	59	1	890605	1.70	091889	4.31	59
2	884466	1.76	076700	4.28	58	2	890707	1.70	092147	4.31	58
3	884572	1.76	076956	4.28	57	3	890809	1.70	092406	4.31	57
4	884677	1.76	077213	4.28	56	4	890911	1.70	092664	4.31	56
5	884783	1.76	077470	4.28	55	5	891013	1.70	092923	4.31	55
6	884889	1.76	077726	4.28	54	6	891115	1.70	093181	4.31	54
7	884994	1.76	077983	4.28	53	7	891217	1.70	093440	4.31	53
8	885100	1.76	078240	4.28	52	8	891319	1.70	093698	4.31	52
9	885205	1.76	078497	4.28	51	9	891421	1.70	093957	4.31	51
10	885311	1.75	078753	4.28	50	10	891523	1.69	094215	4.31	50
11	9.885416	1.75	10.079010	4.28	49	11	9.891624	1.69	10.094474	4.31	49
12	885522	1.75	079267	4.28	48	12	891726	1.69	094733	4.31	48
13	885627	1.75	079524	4.28	47	13	891827	1.69	094992	4.31	47
14	885732	1.75	079781	4.28	46	14	891929	1.69	095250	4.31	46
15	885837	1.75	080038	4.28	45	15	892030	1.69	095509	4.31	45
16	885942	1.75	080295	4.28	44	16	892132	1.69	095768	4.31	44
17	886047	1.75	080552	4.28	43	17	892233	1.69	096027	4.31	43
18	886152	1.75	080809	4.28	42	18	892334	1.69	096286	4.31	42
19	886257	1.75	081036	4.28	41	19	892435	1.68	096544	4.31	41
20	886362	1.74	081323	4.28	40	20	892536	1.68	096803	4.32	40
21	9.886466	1.74	10.081580	4.28	39	21	9.892638	1.68	10.097062	4.32	39
22	886571	1.74	081837	4.29	38	22	892739	1.68	097321	4.32	38
23	886676	1.74	082094	4.29	37	23	892839	1.68	097580	4.32	37
24	886780	1.74	082352	4.29	36	24	892940	1.68	097840	4.32	36
25	886885	1.74	082609	4.29	35	25	893041	1.68	098099	4.32	35
26	886989	1.74	082866	4.29	34	26	893142	1.68	098358	4.32	34
27	887093	1.74	083123	4.29	33	27	893243	1.68	098617	4.32	33
28	887198	1.74	083381	4.29	32	28	893343	1.68	098876	4.32	32
29	887302	1.74	083638	4.29	31	29	893444	1.67	099136	4.32	31
30	887403	1.73	083896	4.29	30	30	893544	1.67	099395	4.32	30
31	9.887510	1.73	10.084153	4.29	29	31	9.893645	1.67	10.099654	4.32	29
32	887614	1.73	084410	4.29	28	32	893745	1.67	099913	4.32	28
33	887718	1.73	084668	4.29	27	33	893846	1.67	100173	4.32	27
34	887822	1.73	084925	4.29	26	34	893946	1.67	100432	4.32	26
35	887925	1.73	085183	4.29	25	35	894046	1.67	100692	4.32	25
36	888030	1.73	085440	4.29	24	36	894146	1.67	100951	4.33	24
37	888134	1.73	085698	4.29	23	37	894246	1.67	101211	4.33	23
38	888237	1.73	085956	4.29	22	38	894346	1.67	101470	4.33	22
39	888341	1.73	086213	4.29	21	39	894446	1.67	101730	4.33	21
40	888444	1.73	086471	4.29	20	40	894546	1.66	101990	4.33	20
41	9.888548	1.72	10.086729	4.29	19	41	9.894646	1.66	10.102249	4.33	19
42	888651	1.72	086986	4.30	18	42	894746	1.66	102509	4.33	18
43	888755	1.72	087244	4.30	17	43	894846	1.66	102769	4.33	17
44	888858	1.72	087502	4.30	16	44	894945	1.66	103029	4.33	16
45	888961	1.72	087760	4.30	15	45	895045	1.66	103288	4.33	15
46	889064	1.72	088018	4.30	14	46	895145	1.66	103548	4.33	14
47	889168	1.72	088275	4.30	13	47	895244	1.66	103808	4.33	13
48	889271	1.72	088533	4.30	12	48	895343	1.66	104068	4.33	12
49	889374	1.72	088791	4.30	11	49	895443	1.65	104328	4.33	11
50	889477	1.71	089049	4.30	10	50	895542	1.65	104588	4.33	10
51	9.889579	1.71	10.089307	4.30	9	51	9.895641	1.65	10.104848	4.33	9
52	889682	1.71	089565	4.30	8	52	895741	1.65	105108	4.33	8
53	889785	1.71	089823	4.30	7	53	895840	1.65	105368	4.34	7
54	889888	1.71	090082	4.30	6	54	895939	1.65	105628	4.34	6
55	889990	1.71	090340	4.30	5	55	896038	1.65	105889	4.34	5
56	890093	1.71	090598	4.30	4	56	896137	1.65	106149	4.34	4
57	890195	1.71	090856	4.30	3	57	896236	1.65	106409	4.34	3
58	890298	1.71	091114	4.30	2	58	896335	1.65	106669	4.34	2
59	890400	1.71	091372	4.30	1	59	896433	1.65	106930	4.34	1
60	890503		091631		0	60	896532		107190		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.896532	1.64	10.107190	4.34	60	0	9.902349	1.59	10.122886	4.38	60
1	896631	1.64	107451	4.34	59	1	902444	1.59	123148	4.38	59
2	896729	1.64	107711	4.34	58	2	902539	1.58	123411	4.38	58
3	896828	1.64	107972	4.34	57	3	902634	1.58	123674	4.38	57
4	896926	1.64	108232	4.34	56	4	902729	1.58	123937	4.38	56
5	897025	1.64	108493	4.34	55	5	902824	1.58	124200	4.38	55
6	897123	1.64	108753	4.34	54	6	902919	1.58	124463	4.38	54
7	897222	1.64	109014	4.34	53	7	903014	1.58	124727	4.39	53
8	897320	1.64	109275	4.35	52	8	903108	1.58	124990	4.39	52
9	897418	1.63	109535	4.35	51	9	903203	1.58	125253	4.39	51
10	897516	1.63	109796	4.35	50	10	903298	1.58	125516	4.39	50
11	9.897614	1.63	10.110057	4.35	49	11	9.903392	1.58	10.125780	4.39	49
12	897712	1.63	110318	4.35	48	12	903487	1.57	126043	4.39	48
13	897810	1.63	110579	4.35	47	13	903581	1.57	126306	4.39	47
14	897908	1.63	110839	4.35	46	14	903676	1.57	126570	4.39	46
15	898006	1.63	111100	4.35	45	15	903770	1.57	126833	4.39	45
16	898104	1.63	111361	4.35	44	16	903864	1.57	127097	4.39	44
17	898202	1.63	111622	4.35	43	17	903959	1.57	127360	4.39	43
18	898299	1.63	111884	4.35	42	18	904053	1.57	127624	4.39	42
19	898397	1.63	112145	4.35	41	19	904147	1.57	127888	4.39	41
20	898494	1.63	112406	4.35	40	20	904241	1.57	128151	4.40	40
21	9.898592	1.62	10.112667	4.35	39	21	9.904335	1.57	10.128415	4.40	39
22	898689	1.62	112928	4.35	38	22	904429	1.57	128679	4.40	38
23	898787	1.62	113189	4.35	37	23	904523	1.56	128943	4.40	37
24	898884	1.62	113451	4.36	36	24	904617	1.56	129207	4.40	36
25	898981	1.62	113712	4.36	35	25	904711	1.56	129471	4.40	35
26	899078	1.62	113974	4.36	34	26	904804	1.56	129735	4.40	34
27	899176	1.62	114235	4.36	33	27	904898	1.56	129999	4.40	33
28	899273	1.62	114496	4.36	32	28	904992	1.56	130263	4.40	32
29	899370	1.62	114758	4.36	31	29	905085	1.56	130527	4.40	31
30	899467	1.62	115020	4.36	30	30	905179	1.56	130791	4.40	30
31	9.899564	1.61	10.115281	4.36	29	31	9.905272	1.56	10.131055	4.40	29
32	899660	1.61	115543	4.36	28	32	905366	1.55	131320	4.40	28
33	899757	1.61	115804	4.36	27	33	905459	1.55	131584	4.40	27
34	899854	1.61	116066	4.36	26	34	905552	1.55	131848	4.41	26
35	899951	1.61	116328	4.36	25	35	905645	1.55	132113	4.41	25
36	900047	1.61	116590	4.36	24	36	905739	1.55	132377	4.41	24
37	900144	1.61	116852	4.36	23	37	905832	1.55	132642	4.41	23
38	900240	1.61	117113	4.36	22	38	905925	1.55	132906	4.41	22
39	900337	1.61	117375	4.37	21	39	906018	1.55	133171	4.41	21
40	900433	1.60	117637	4.37	20	40	906111	1.55	133436	4.41	20
41	9.900529	1.60	10.117899	4.37	19	41	9.906204	1.55	10.133700	4.41	19
42	900626	1.60	118161	4.37	18	42	906296	1.55	133965	4.41	18
43	900722	1.60	118423	4.37	17	43	906389	1.55	134230	4.41	17
44	900818	1.60	118686	4.37	16	44	906482	1.54	134495	4.41	16
45	900914	1.60	118948	4.37	15	45	906575	1.54	134760	4.42	15
46	901010	1.60	119210	4.37	14	46	906667	1.54	135025	4.42	14
47	901106	1.60	119472	4.37	13	47	906760	1.54	135290	4.42	13
48	901202	1.60	119735	4.37	12	48	906852	1.54	135555	4.42	12
49	901298	1.60	119997	4.37	11	49	906945	1.54	135820	4.42	11
50	901394	1.60	120259	4.37	10	50	907037	1.54	136085	4.42	10
51	9.901490	1.59	10.120522	4.37	9	51	9.907129	1.54	10.136350	4.42	9
52	901585	1.59	120784	4.38	8	52	907222	1.54	136615	4.42	8
53	901681	1.59	121047	4.38	7	53	907314	1.53	136881	4.42	7
54	901776	1.59	121309	4.38	6	54	907406	1.53	137146	4.42	6
55	901872	1.59	121572	4.38	5	55	907498	1.53	137411	4.42	5
56	901967	1.59	121835	4.38	4	56	907590	1.53	137677	4.42	4
57	902063	1.59	122097	4.38	3	57	907682	1.53	137942	4.43	3
58	902158	1.59	122360	4.38	2	58	907774	1.53	138208	4.43	2
59	902253	1.59	122623	4.38	1	59	907866	1.53	138473	4.43	1
60	902349		122886		0	60	907958		138739		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.907958	1.53	10.138739	4.43	60	0	9.913365	1.47	10.154773	4.48	60
1	908049	1.53	139005	4.43	59	1	913453	1.47	155042	4.48	59
2	908141	1.53	139270	4.43	58	2	913541	1.47	155311	4.48	58
3	908233	1.53	139536	4.43	57	3	913630	1.47	155580	4.48	57
4	908324	1.53	139802	4.43	56	4	913718	1.47	155849	4.48	56
5	908416	1.52	140068	4.43	55	5	913806	1.47	156118	4.49	55
6	908507	1.52	140334	4.43	54	6	913894	1.47	156388	4.49	54
7	908599	1.52	140600	4.43	53	7	913982	1.47	156657	4.49	53
8	908690	1.52	140866	4.43	52	8	914070	1.47	156926	4.49	52
9	908781	1.52	141132	4.43	51	9	914158	1.47	157195	4.49	51
10	908873	1.52	141398	4.44	50	10	914246	1.47	157465	4.49	50
11	9.908964	1.52	10.141664	4.44	49	11	9.914334	1.46	10.157734	4.49	49
12	909055	1.52	141931	4.44	48	12	914422	1.46	158004	4.49	48
13	909146	1.52	142197	4.44	47	13	914510	1.46	158273	4.49	47
14	909237	1.52	142463	4.44	46	14	914598	1.46	158543	4.49	46
15	909328	1.52	142730	4.44	45	15	914685	1.46	158813	4.50	45
16	909419	1.51	142996	4.44	44	16	914773	1.46	159083	4.50	44
17	909510	1.51	143263	4.44	43	17	914860	1.46	159352	4.50	43
18	909601	1.51	143529	4.44	42	18	914948	1.46	159622	4.50	42
19	909691	1.51	143796	4.44	41	19	915035	1.46	159892	4.50	41
20	909782	1.51	144062	4.45	40	20	915123	1.45	160162	4.50	40
21	9.909873	1.51	10.144329	4.45	39	21	9.915210	1.45	10.160432	4.50	39
22	909963	1.51	144596	4.45	38	22	915297	1.45	160703	4.50	38
23	910054	1.51	144863	4.45	37	23	915385	1.45	160973	4.50	37
24	910144	1.51	145130	4.45	36	24	915472	1.45	161243	4.50	36
25	910235	1.51	145397	4.45	35	25	915559	1.45	161513	4.51	35
26	910325	1.50	145664	4.45	34	26	915646	1.45	161784	4.51	34
27	910415	1.50	145931	4.45	33	27	915733	1.45	162054	4.51	33
28	910506	1.50	146198	4.45	32	28	915820	1.45	162325	4.51	32
29	910596	1.50	146465	4.45	31	29	915907	1.45	162595	4.51	31
30	910686	1.50	146732	4.45	30	30	915994	1.45	162866	4.51	30
31	9.910776	1.50	10.146999	4.46	29	31	9.916081	1.45	10.163136	4.51	29
32	910866	1.50	147267	4.46	28	32	916167	1.45	163407	4.51	28
33	910956	1.50	147534	4.46	27	33	916254	1.44	163678	4.51	27
34	911046	1.50	147801	4.46	26	34	916341	1.44	163949	4.52	26
35	911136	1.50	148069	4.46	25	35	916427	1.44	164220	4.52	25
36	911226	1.50	148336	4.46	24	36	916514	1.44	164491	4.52	24
37	911315	1.50	148604	4.46	23	37	916600	1.44	164762	4.52	23
38	911405	1.49	148871	4.46	22	38	916687	1.44	165033	4.52	22
39	911495	1.49	149139	4.46	21	39	916773	1.44	165304	4.52	21
40	911584	1.49	149407	4.46	20	40	916859	1.44	165575	4.52	20
41	9.911674	1.49	10.149675	4.46	19	41	9.916946	1.44	10.165846	4.52	19
42	911763	1.49	149943	4.47	18	42	917032	1.44	166118	4.52	18
43	911853	1.49	150210	4.47	17	43	917118	1.43	166389	4.52	17
44	911942	1.49	150478	4.47	16	44	917204	1.43	166661	4.53	16
45	912031	1.49	150746	4.47	15	45	917290	1.43	166932	4.53	15
46	912121	1.49	151014	4.47	14	46	917376	1.43	167204	4.53	14
47	912210	1.49	151283	4.47	13	47	917462	1.43	167475	4.53	13
48	912299	1.48	151551	4.47	12	48	917548	1.43	167747	4.53	12
49	912388	1.48	151819	4.47	11	49	917634	1.43	168019	4.53	11
50	912477	1.48	152087	4.47	10	50	917719	1.43	168291	4.53	10
51	9.912566	1.48	10.152356	4.47	9	51	9.917805	1.43	10.168563	4.53	9
52	912655	1.48	152624	4.47	8	52	917891	1.43	168835	4.53	8
53	912744	1.48	152892	4.47	7	53	917976	1.42	169107	4.53	7
54	912833	1.48	153161	4.48	6	54	918062	1.42	169379	4.53	6
55	912922	1.48	153430	4.48	5	55	918147	1.42	169651	4.54	5
56	913010	1.48	153698	4.48	4	56	918233	1.42	169923	4.54	4
57	913099	1.48	153967	4.48	3	57	918318	1.42	170195	4.54	3
58	913187	1.48	154236	4.48	2	58	918404	1.42	170468	4.54	2
59	913276	1.47	154504	4.48	1	59	918489	1.42	170740	4.54	1
60	913365		154773		0	60	918574		171013		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.918574	1.42	10.171013	4.54	60	0	9.923591	1.37	10.187483	4.61	60
1	918659	1.42	171285	4.54	59	1	923673	1.37	187759	4.61	59
2	918745	1.42	171558	4.54	58	2	923755	1.37	188036	4.61	58
3	918830	1.42	171830	4.55	57	3	923837	1.36	188313	4.61	57
4	918915	1.42	172103	4.55	56	4	923919	1.36	188590	4.61	56
5	919000	1.41	172376	4.55	55	5	924001	1.36	188866	4.62	55
6	919085	1.41	172649	4.55	54	6	924083	1.36	189143	4.62	54
7	919169	1.41	172922	4.55	53	7	924164	1.36	189420	4.62	53
8	919254	1.41	173195	4.55	52	8	924246	1.36	189698	4.62	52
9	919339	1.41	173468	4.55	51	9	924328	1.36	189975	4.62	51
10	919424	1.41	173741	4.55	50	10	924409	1.36	190252	4.62	50
11	9.919508	1.41	10.174014	4.55	49	11	9.924491	1.36	10.190529	4.62	49
12	919593	1.41	174287	4.56	48	12	924572	1.36	190807	4.62	48
13	919677	1.41	174561	4.56	47	13	924654	1.36	191084	4.63	47
14	919762	1.41	174834	4.56	46	14	924735	1.35	191362	4.63	46
15	919846	1.41	175107	4.56	45	15	924816	1.35	191639	4.63	45
16	919931	1.40	175381	4.56	44	16	924897	1.35	191917	4.63	44
17	920015	1.40	175655	4.56	43	17	924979	1.35	192195	4.63	43
18	920099	1.40	175928	4.56	42	18	925060	1.35	192473	4.63	42
19	920184	1.40	176202	4.56	41	19	925141	1.35	192751	4.63	41
20	920268	1.40	176476	4.56	40	20	925222	1.35	193029	4.63	40
21	9.920352	1.40	10.176749	4.56	39	21	9.925303	1.35	10.193307	4.63	39
22	920436	1.40	177023	4.57	38	22	925384	1.35	193585	4.64	38
23	920520	1.40	177297	4.57	37	23	925465	1.35	193863	4.64	37
24	920604	1.40	177571	4.57	36	24	925545	1.35	194141	4.64	36
25	920688	1.40	177846	4.57	35	25	925626	1.34	194420	4.64	35
26	920772	1.40	178120	4.57	34	26	925707	1.34	194698	4.64	34
27	920856	1.40	178394	4.57	33	27	925788	1.34	194977	4.64	33
28	920939	1.40	178668	4.57	32	28	925868	1.34	195255	4.65	32
29	921023	1.39	178943	4.57	31	29	925949	1.34	195534	4.65	31
30	921107	1.39	179217	4.58	30	30	926029	1.34	195813	4.65	30
31	9.921190	1.39	10.179492	4.58	29	31	9.926110	1.34	10.196091	4.65	29
32	921274	1.39	179766	4.58	28	32	926190	1.34	196370	4.65	28
33	921357	1.39	180041	4.58	27	33	926270	1.34	196649	4.65	27
34	921441	1.39	180316	4.58	26	34	926351	1.34	196928	4.65	26
35	921524	1.39	180590	4.58	25	35	926431	1.34	197208	4.65	25
36	921607	1.39	180865	4.58	24	36	926511	1.33	197487	4.65	24
37	921691	1.39	181140	4.58	23	37	926591	1.33	197766	4.66	23
38	921774	1.39	181415	4.58	22	38	926671	1.33	198045	4.66	22
39	921857	1.38	181690	4.59	21	39	926751	1.33	198325	4.66	21
40	921940	1.38	181965	4.59	20	40	926831	1.33	198604	4.66	20
41	9.922023	1.38	10.182241	4.59	19	41	9.926911	1.33	10.198884	4.66	19
42	922106	1.38	182516	4.59	18	42	926991	1.33	199164	4.66	18
43	922189	1.38	182791	4.59	17	43	927071	1.33	199443	4.66	17
44	922272	1.38	183067	4.59	16	44	927151	1.33	199723	4.67	16
45	922355	1.38	183342	4.59	15	45	927231	1.33	200003	4.67	15
46	922438	1.38	183618	4.59	14	46	927310	1.33	200283	4.67	14
47	922520	1.38	183893	4.60	13	47	927390	1.33	200563	4.67	13
48	922603	1.38	184169	4.60	12	48	927470	1.32	200843	4.67	12
49	922686	1.38	184445	4.60	11	49	927549	1.32	201123	4.67	11
50	922768	1.38	184720	4.60	10	50	927629	1.32	201404	4.67	10
51	9.922851	1.37	10.184996	4.60	9	51	9.927708	1.32	10.201684	4.67	9
52	922933	1.37	185272	4.60	8	52	927787	1.32	201964	4.68	8
53	923016	1.37	185548	4.60	7	53	927867	1.32	202245	4.68	7
54	923098	1.37	185824	4.60	6	54	927946	1.32	202526	4.68	6
55	923181	1.37	186101	4.60	5	55	928025	1.32	202806	4.68	5
56	923263	1.37	186377	4.60	4	56	928104	1.32	203087	4.68	4
57	923345	1.37	186653	4.61	3	57	928183	1.32	203368	4.68	3
58	923427	1.37	186930	4.61	2	58	928263	1.32	203649	4.68	2
59	923509	1.37	187206	4.61	1	59	928342	1.32	203930	4.68	1
60	923591		187483		0	60	928420		204211		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.928420	1.31	10.204211	4.68	60	0	9.933066	1.26	10.221226	4.77	60
1	928499	1.31	204492	4.69	59	1	933141	1.26	221512	4.77	59
2	928578	1.31	204773	4.69	58	2	933217	1.26	221799	4.77	58
3	928657	1.31	205054	4.69	57	3	933293	1.26	222085	4.77	57
4	928736	1.31	205336	4.69	56	4	933369	1.26	222372	4.78	56
5	928815	1.31	205617	4.69	55	5	933445	1.26	222658	4.78	55
6	928893	1.31	205899	4.69	54	6	933520	1.26	222945	4.78	54
7	928972	1.31	206181	4.69	53	7	933596	1.26	223232	4.78	53
8	929050	1.31	206462	4.70	52	8	933671	1.26	223518	4.78	52
9	929129	1.31	206744	4.70	51	9	933747	1.26	223805	4.78	51
10	929207	1.31	207026	4.70	50	10	933822	1.26	224092	4.78	50
11	9.929286	1.31	10.207308	4.70	49	11	9.933898	1.25	10.224379	4.79	49
12	929364	1.30	207590	4.70	48	12	933973	1.25	224667	4.79	48
13	929442	1.30	207872	4.70	47	13	934048	1.25	224954	4.79	47
14	929521	1.30	208154	4.70	46	14	934123	1.25	225241	4.79	46
15	929599	1.30	208437	4.70	45	15	934199	1.25	225529	4.79	45
16	929677	1.30	208719	4.71	44	16	934274	1.25	225816	4.79	44
17	929755	1.30	209001	4.71	43	17	934349	1.25	22610.	4.79	43
18	929833	1.30	209284	4.71	42	18	934424	1.25	226392	4.80	42
19	929911	1.30	209566	4.71	41	19	934499	1.25	226679	4.80	41
20	929989	1.30	209849	4.71	40	20	934574	1.25	226967	4.80	40
21	9.930067	1.30	10.210132	4.71	39	21	9.934649	1.25	10.227255	4.80	39
22	930145	1.30	210415	4.72	38	22	934723	1.25	227543	4.80	38
23	930223	1.30	210698	4.72	37	23	934798	1.25	227832	4.80	37
24	930300	1.29	210981	4.72	36	24	934873	1.24	228120	4.80	36
25	930378	1.29	211264	4.72	35	25	934948	1.24	228408	4.81	35
26	930456	1.29	211547	4.72	34	26	935022	1.24	228697	4.81	34
27	930533	1.29	211830	4.72	33	27	935097	1.24	228985	4.81	33
28	930611	1.29	212114	4.72	32	28	935171	1.24	229274	4.81	32
29	930688	1.29	212397	4.72	31	29	935246	1.24	229563	4.81	31
30	930766	1.29	212681	4.73	30	30	935320	1.24	229852	4.81	30
31	9.930843	1.29	10.212964	4.73	29	31	9.935395	1.24	10.230140	4.82	29
32	930921	1.29	213248	4.73	28	32	935469	1.24	230429	4.82	28
33	930998	1.29	213532	4.73	27	33	935543	1.24	230719	4.82	27
34	931075	1.29	213816	4.73	26	34	935618	1.24	231008	4.82	26
35	931152	1.29	214100	4.73	25	35	935692	1.23	231297	4.82	25
36	931229	1.28	214384	4.73	24	36	935766	1.23	231586	4.82	24
37	931306	1.28	214668	4.73	23	37	935840	1.23	231876	4.83	23
38	931383	1.28	214952	4.74	22	38	935914	1.23	232166	4.83	22
39	931460	1.28	215236	4.74	21	39	935988	1.23	232455	4.83	21
40	931537	1.28	215521	4.74	20	40	936062	1.23	232745	4.83	20
41	9.931614	1.28	10.215805	4.74	19	41	9.936136	1.23	10.233035	4.83	19
42	931691	1.28	216090	4.74	18	42	936210	1.23	233325	4.83	18
43	931768	1.28	216374	4.75	17	43	936284	1.23	233615	4.83	17
44	931845	1.28	216659	4.75	16	44	936357	1.23	233905	4.84	16
45	931921	1.28	216944	4.75	15	45	936431	1.23	234195	4.84	15
46	931998	1.28	217229	4.75	14	46	936505	1.23	234486	4.84	14
47	932075	1.27	217514	4.75	13	47	936578	1.23	234776	4.84	13
48	932151	1.27	217799	4.75	12	48	936652	1.22	235067	4.84	12
49	932228	1.27	218084	4.75	11	49	936725	1.22	235357	4.85	11
50	932304	1.27	218369	4.75	10	50	936799	1.22	235648	4.85	10
51	9.932380	1.27	10.218654	4.76	9	51	9.936872	1.22	10.235939	4.85	9
52	932457	1.27	218940	4.76	8	52	936946	1.22	236230	4.85	8
53	932533	1.27	219225	4.76	7	53	937019	1.22	236521	4.85	7
54	932609	1.27	219511	4.76	6	54	937092	1.22	236812	4.85	6
55	932685	1.27	219797	4.76	5	55	937165	1.22	237103	4.85	5
56	932762	1.27	220082	4.76	4	56	937238	1.22	237394	4.86	4
57	932838	1.27	220368	4.77	3	57	937312	1.22	237686	4.86	3
58	932914	1.27	220654	4.77	2	58	937385	1.22	237977	4.86	2
59	932990	1.27	220940	4.77	1	59	937458	1.22	238269	4.86	1
60	933066		221226		0	60	937531		238561		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

60°

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.937531	1.21	10.238561	4.86	60
1	937604	1.21	238852	4.86	59
2	937676	1.21	239144	4.87	58
3	937749	1.21	239436	4.87	57
4	937822	1.21	239728	4.87	56
5	937895	1.21	240021	4.87	55
6	937967	1.21	240313	4.87	54
7	938040	1.21	240605	4.87	53
8	938113	1.21	240898	4.88	52
9	938185	1.21	241190	4.88	51
10	938258	1.21	241483	4.88	50
11	9.938330	1.21	10.241776	4.88	49
12	938402	1.20	242069	4.88	48
13	938475	1.20	242362	4.88	47
14	938547	1.20	242655	4.89	46
15	938619	1.20	242948	4.89	45
16	938691	1.20	243241	4.89	44
17	938763	1.20	243535	4.89	43
18	938836	1.20	243828	4.89	42
19	938908	1.20	244122	4.89	41
20	938980	1.20	244415	4.90	40
21	9.939052	1.20	10.244709	4.90	39
22	939123	1.20	245003	4.90	38
23	939195	1.20	245297	4.90	37
24	939267	1.19	245591	4.90	36
25	939339	1.19	245885	4.90	35
26	939410	1.19	246180	4.91	34
27	939182	1.19	246474	4.91	33
28	939554	1.19	246769	4.91	32
29	939625	1.19	247063	4.91	31
30	939697	1.19	247358	4.91	30
31	9.939768	1.19	10.247653	4.91	29
32	939840	1.19	247948	4.92	28
33	939911	1.19	248243	4.92	27
34	939982	1.19	248538	4.92	26
35	940054	1.19	248833	4.92	25
36	940125	1.18	249128	4.92	24
37	940196	1.18	249424	4.92	23
38	940267	1.18	249719	4.93	22
39	940338	1.18	250015	4.93	21
40	940409	1.18	250311	4.93	20
41	9.940480	1.18	10.250607	4.93	19
42	940551	1.18	250903	4.93	18
43	940322	1.18	251199	4.93	17
44	940693	1.18	251495	4.94	16
45	940763	1.18	251791	4.94	15
46	940834	1.18	252087	4.94	14
47	940905	1.18	252384	4.94	13
48	940975	1.18	252681	4.94	12
49	941046	1.17	252977	4.95	11
50	941117	1.17	253274	4.95	10
51	9.941187	1.17	10.253571	4.95	9
52	941258	1.17	253868	4.95	8
53	941328	1.17	254165	4.95	7
54	941398	1.17	254462	4.96	6
55	941469	1.17	254760	4.96	5
56	941539	1.17	255057	4.96	4
57	941609	1.17	255355	4.96	3
58	941679	1.17	255652	4.96	2
59	941749	1.17	255950	4.96	1
60	941819		256248		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

29°

61°

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.941819	1.17	10.256248	4.97	60
1	941889	1.16	256546	4.97	59
2	941959	1.16	256844	4.97	58
3	942029	1.16	257142	4.97	57
4	942099	1.16	257441	4.97	56
5	942169	1.16	257739	4.97	55
6	942239	1.16	258038	4.98	54
7	942308	1.16	258336	4.98	53
8	942378	1.16	258635	4.98	52
9	942448	1.16	258934	4.98	51
10	942517	1.16	259233	4.98	50
11	9.942587	1.16	10.259532	4.99	49
12	942656	1.16	259831	4.99	48
13	942726	1.16	260130	4.99	47
14	942795	1.15	260430	4.99	46
15	942864	1.15	260729	4.99	45
16	942934	1.15	261029	4.99	44
17	943003	1.15	261329	5.00	43
18	943072	1.15	261629	5.00	42
19	943141	1.15	261929	5.00	41
20	943210	1.15	262229	5.00	40
21	9.943279	1.15	10.262529	5.00	39
22	943348	1.15	262829	5.01	38
23	943417	1.15	263130	5.01	37
24	943486	1.15	263430	5.01	36
25	943555	1.15	263731	5.01	35
26	943624	1.15	264031	5.01	34
27	943693	1.14	264332	5.02	33
28	943761	1.14	264633	5.02	32
29	943830	1.14	264934	5.02	31
30	943899	1.14	265236	5.02	30
31	9.943967	1.14	10.265537	5.02	29
32	944036	1.14	265838	5.02	28
33	944104	1.14	266140	5.03	27
34	944172	1.14	266442	5.03	26
35	944241	1.14	266743	5.03	25
36	944309	1.14	267045	5.03	24
37	944377	1.14	267347	5.03	23
38	944446	1.14	267649	5.04	22
39	944514	1.14	267952	5.04	21
40	944582	1.13	268254	5.04	20
41	9.944650	1.13	10.268556	5.04	19
42	944718	1.13	268859	5.04	18
43	944786	1.13	269162	5.05	17
44	944854	1.13	269465	5.05	16
45	944922	1.13	269767	5.05	15
46	944990	1.13	270071	5.05	14
47	945058	1.13	270374	5.05	13
48	945125	1.13	270677	5.06	12
49	945193	1.13	270980	5.06	11
50	945261	1.13	271284	5.06	10
51	9.945328	1.13	10.271588	5.06	9
52	945396	1.13	271891	5.06	8
53	945464	1.12	272195	5.07	7
54	945531	1.12	272499	5.07	6
55	945598	1.12	272803	5.07	5
56	945666	1.12	273108	5.07	4
57	945733	1.12	273412	5.07	3
58	945800	1.12	273716	5.08	2
59	945868	1.12	274021	5.08	1
60	945935		274326		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

28°

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.945935	1.12	10.274326	5.08	60	0	9.949881	1.07	10.292834	5.21	60
1	946002	1.12	274630	5.08	59	1	949945	1.07	293146	5.21	59
2	946069	1.12	274935	5.08	58	2	950010	1.07	293459	5.21	58
3	946136	1.12	275240	5.09	57	3	950074	1.07	293772	5.21	57
4	946203	1.12	275546	5.09	56	4	950138	1.07	294084	5.21	56
5	946270	1.11	275851	5.09	55	5	950202	1.07	294397	5.22	55
6	946337	1.11	276156	5.09	54	6	950266	1.07	294710	5.22	54
7	946404	1.11	276462	5.09	53	7	950330	1.07	295024	5.22	53
8	946471	1.11	276768	5.10	52	8	950394	1.07	295337	5.22	52
9	946538	1.11	277073	5.10	51	9	950458	1.07	295650	5.22	51
10	946604	1.11	277379	5.10	50	10	950522	1.06	295964	5.23	50
11	9.946671	1.11	10.277685	5.10	49	11	9.950586	1.06	10.296278	5.23	49
12	946738	1.11	277991	5.10	48	12	950650	1.06	296591	5.23	48
13	946804	1.11	278298	5.11	47	13	950714	1.06	296905	5.23	47
14	946871	1.11	278604	5.11	46	14	950778	1.06	297219	5.24	46
15	946937	1.11	278911	5.11	45	15	950841	1.06	297534	5.24	45
16	947004	1.11	279217	5.11	44	16	950905	1.06	297848	5.24	44
17	947070	1.11	279524	5.11	43	17	950968	1.06	298163	5.24	43
18	947136	1.10	279831	5.12	42	18	951032	1.06	298477	5.24	42
19	947203	1.10	280138	5.12	41	19	951096	1.06	298792	5.25	41
20	947269	1.10	280445	5.12	40	20	951159	1.06	299107	5.25	40
21	9.947335	1.10	10.280752	5.12	39	21	9.951222	1.06	10.299422	5.25	39
22	947401	1.10	281060	5.12	38	22	951286	1.06	299737	5.26	38
23	947467	1.10	281367	5.13	37	23	951349	1.05	300053	5.26	37
24	947533	1.10	281675	5.13	36	24	951412	1.05	300368	5.26	36
25	947600	1.10	281983	5.13	35	25	951476	1.05	300684	5.26	35
26	947665	1.10	282291	5.13	34	26	951539	1.05	300999	5.26	34
27	947731	1.10	282599	5.13	33	27	951602	1.05	301315	5.27	33
28	947797	1.10	282907	5.14	32	28	951665	1.05	301631	5.27	32
29	947863	1.10	283215	5.14	31	29	951728	1.05	301947	5.27	31
30	947929	1.10	283523	5.14	30	30	951791	1.05	302264	5.27	30
31	9.947995	1.09	10.283832	5.14	29	31	9.951854	1.05	10.302580	5.28	29
32	948060	1.09	284140	5.14	28	32	951917	1.05	302897	5.28	28
33	948126	1.09	284449	5.15	27	33	951980	1.05	303213	5.28	27
34	948192	1.09	284758	5.15	26	34	952043	1.05	303530	5.28	26
35	948257	1.09	285067	5.15	25	35	952106	1.05	303847	5.29	25
36	948323	1.09	285376	5.15	24	36	952168	1.04	304164	5.29	24
37	948388	1.09	285686	5.16	23	37	952231	1.04	304482	5.29	23
38	948454	1.09	285995	5.16	22	38	952294	1.04	304799	5.29	22
39	948519	1.09	286304	5.16	21	39	952356	1.04	305117	5.29	21
40	948584	1.09	286614	5.16	20	40	952419	1.04	305434	5.30	20
41	9.948650	1.09	10.286924	5.16	19	41	9.952481	1.04	10.305752	5.30	19
42	948715	1.09	287234	5.17	18	42	952544	1.04	306070	5.30	18
43	948780	1.08	287544	5.17	17	43	952606	1.04	306388	5.30	17
44	948845	1.08	287854	5.17	16	44	952669	1.04	306707	5.31	16
45	948910	1.08	288164	5.17	15	45	952731	1.04	307025	5.31	15
46	948975	1.08	288475	5.18	14	46	952793	1.04	307344	5.31	14
47	949040	1.08	288785	5.18	13	47	952855	1.04	307662	5.31	13
48	949105	1.08	289096	5.18	12	48	952918	1.04	307981	5.31	12
49	949170	1.08	289407	5.18	11	49	952980	1.03	308300	5.32	11
50	949235	1.08	289718	5.18	10	50	953042	1.03	308619	5.32	10
51	9.949300	1.08	10.290029	5.19	9	51	9.953104	1.03	10.308938	5.32	9
52	949364	1.08	290340	5.19	8	52	953166	1.03	309258	5.33	8
53	949429	1.08	290651	5.19	7	53	953228	1.03	309577	5.33	7
54	949494	1.08	290963	5.19	6	54	953290	1.03	309897	5.33	6
55	949558	1.08	291274	5.19	5	55	953352	1.03	310217	5.33	5
56	949623	1.08	291586	5.20	4	56	953413	1.03	310537	5.33	4
57	949688	1.07	291898	5.20	3	57	953475	1.03	310857	5.34	3
58	949752	1.07	292210	5.20	2	58	953537	1.03	311177	5.34	2
59	949816	1.07	292522	5.20	1	59	953599	1.03	311498	5.34	1
60	949881		292834		0	60	953660		311818		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.953660	1.03	10.311818	5.34	60	0	9.957276	.98	10.331327	5.50	60
1	953722	1.02	312139	5.35	59	1	957335	.98	331657	5.50	59
2	953783	1.02	312460	5.35	58	2	957393	.98	331987	5.50	58
3	953845	1.02	312781	5.35	57	3	957452	.98	332318	5.51	57
4	953906	1.02	313102	5.35	56	4	957511	.98	332648	5.51	56
5	953968	1.02	313423	5.36	55	5	957570	.98	332979	5.51	55
6	954029	1.02	313745	5.36	54	6	957628	.98	333309	5.51	54
7	954090	1.02	314066	5.36	53	7	957687	.98	333640	5.52	53
8	954152	1.02	314388	5.36	52	8	957746	.97	333971	5.52	52
9	954213	1.02	314710	5.37	51	9	957804	.97	334302	5.52	51
10	954274	1.02	315032	5.37	50	10	957863	.97	334634	5.53	50
11	9.954335	1.02	10.315354	5.37	49	11	9.957921	.97	10.334965	5.53	49
12	954396	1.02	315676	5.37	48	12	957979	.97	335297	5.53	48
13	954457	1.02	315999	5.38	47	13	958038	.97	335629	5.53	47
14	954518	1.01	316321	5.38	46	14	958096	.97	335961	5.54	46
15	954579	1.01	316644	5.38	45	15	958154	.97	336293	5.54	45
16	954640	1.01	316967	5.38	44	16	958213	.97	336625	5.54	44
17	954701	1.01	317290	5.39	43	17	958271	.97	336958	5.54	43
18	954762	1.01	317613	5.39	42	18	958329	.97	337291	5.55	42
19	954823	1.01	317937	5.39	41	19	958387	.97	337624	5.55	41
20	954883	1.01	318260	5.39	40	20	958445	.97	337957	5.55	40
21	9.954944	1.01	10.318584	5.40	39	21	9.958503	.96	10.338290	5.56	39
22	955005	1.01	318908	5.40	38	22	958561	.96	338623	5.56	38
23	955065	1.01	319232	5.40	37	23	958619	.96	338957	5.56	37
24	955126	1.01	319556	5.40	36	24	958677	.96	339290	5.57	36
25	955186	1.01	319880	5.41	35	25	958734	.96	339624	5.57	35
26	955247	1.01	320205	5.41	34	26	958792	.96	339958	5.57	34
27	955307	1.01	320529	5.41	33	27	958850	.96	340292	5.57	33
28	955368	1.01	320854	5.41	32	28	958908	.96	340627	5.58	32
29	955428	1.00	321179	5.42	31	29	958965	.96	340961	5.58	31
30	955488	1.00	321504	5.42	30	30	959023	.96	341296	5.58	30
31	9.955548	1.00	10.321829	5.42	29	31	9.959080	.96	10.341631	5.58	29
32	955609	1.00	322154	5.42	28	32	959138	.96	341966	5.59	28
33	955669	1.00	322480	5.43	27	33	959195	.96	342301	5.59	27
34	955729	1.00	322806	5.43	26	34	959253	.96	342636	5.59	26
35	955789	1.00	323131	5.43	25	35	959310	.95	342972	5.59	25
36	955849	1.00	323457	5.43	24	36	959368	.95	343308	5.60	24
37	955909	1.00	323783	5.44	23	37	959425	.95	343644	5.60	23
38	955969	1.00	324110	5.44	22	38	959482	.95	343980	5.60	22
39	956029	1.00	324436	5.44	21	39	959539	.95	344316	5.61	21
40	956089	1.00	324763	5.44	20	40	959596	.95	344652	5.61	20
41	9.956148	1.00	10.325089	5.45	19	41	9.959654	.95	10.344989	5.61	19
42	956208	.99	325416	5.45	18	42	959711	.95	345326	5.61	18
43	956268	.99	325743	5.45	17	43	959768	.95	345663	5.62	17
44	956327	.99	326071	5.46	16	44	959825	.95	346000	5.62	16
45	956387	.99	326398	5.46	15	45	959882	.95	346337	5.62	15
46	956447	.99	326726	5.46	14	46	959938	.95	346674	5.63	14
47	956506	.99	327053	5.46	13	47	959995	.95	347012	5.63	13
48	956566	.99	327381	5.47	12	48	960052	.95	347350	5.63	12
49	956625	.99	327709	5.47	11	49	960109	.94	347688	5.63	11
50	956684	.99	328037	5.47	10	50	960165	.94	348026	5.64	10
51	9.956744	.99	10.328365	5.47	9	51	9.960222	.94	10.348364	5.64	9
52	956803	.99	328694	5.48	8	52	960279	.94	348703	5.64	8
53	956862	.99	329023	5.48	7	53	960335	.94	349041	5.65	7
54	956921	.98	329351	5.48	6	54	960392	.94	349380	5.65	6
55	956981	.98	329680	5.48	5	55	960448	.94	349719	5.65	5
56	957040	.98	330009	5.49	4	56	960505	.94	350058	5.66	4
57	957099	.98	330339	5.49	3	57	960561	.94	350398	5.66	3
58	957158	.98	330668	5.49	2	58	960618	.94	350737	5.66	2
59	957217	.98	330998	5.50	1	59	960674	.94	351077	5.66	1
60	957276		331327		0	60	960730		351417		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1'	Cotang.	PP1"	M.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.960730	.94	10.351417	5.67	60	0	9.964026	.89	10.372148	5.85	60
1	960786	.91	351757	5.67	59	1	964080	.89	372499	5.86	59
2	960843	.93	352097	5.67	58	2	964133	.89	372851	5.86	58
3	960899	.93	352438	5.68	57	3	964187	.89	373203	5.86	57
4	960955	.93	352778	5.68	56	4	964240	.89	373555	5.87	56
5	961011	.93	353119	5.68	55	5	964294	.89	373907	5.87	55
6	961067	.93	353460	5.69	54	6	964347	.89	374259	5.87	54
7	961123	.93	353801	5.69	53	7	964400	.89	374612	5.88	53
8	961179	.93	354143	5.69	52	8	964454	.89	374964	5.88	52
9	961235	.93	354484	5.69	51	9	964507	.89	375317	5.88	51
10	961290	.93	354826	5.70	50	10	964560	.89	375670	5.89	50
11	9.961346	.93	10.355168	5.70	49	11	9.964613	.89	10.376024	5.89	49
12	961402	.93	355510	5.70	48	12	964666	.88	376377	5.89	48
13	961458	.93	355852	5.71	47	13	964720	.88	376731	5.90	47
14	961513	.93	356194	5.71	46	14	964773	.88	377085	5.90	46
15	961569	.93	356537	5.71	45	15	964826	.88	377439	5.90	45
16	961624	.92	356880	5.72	44	16	964879	.88	377793	5.91	44
17	961680	.92	357223	5.72	43	17	964931	.88	378148	5.91	43
18	961735	.92	357566	5.72	42	18	964984	.88	378503	5.92	42
19	961791	.92	357909	5.72	41	19	965037	.88	378858	5.92	41
20	961846	.92	358253	5.73	40	20	965090	.88	379213	5.92	40
21	9.961902	.92	10.358596	5.73	39	21	9.965143	.88	10.379568	5.93	39
22	961957	.92	358940	5.73	38	22	965195	.88	379924	5.93	38
23	962012	.92	359284	5.74	37	23	965248	.88	380280	5.93	37
24	962067	.92	359629	5.74	36	24	965301	.88	380636	5.94	36
25	962123	.92	359973	5.74	35	25	965353	.87	380992	5.94	35
26	962178	.92	360318	5.75	34	26	965406	.87	381348	5.94	34
27	962233	.92	360663	5.75	33	27	965458	.87	381705	5.95	33
28	962288	.92	361008	5.75	32	28	965511	.87	382061	5.95	32
29	962343	.92	361353	5.76	31	29	965563	.87	382418	5.95	31
30	962398	.91	361698	5.76	30	30	965615	.87	382776	5.96	30
31	9.962453	.91	10.362044	5.76	29	31	9.965668	.87	10.383133	5.96	29
32	962508	.91	362389	5.77	28	32	965720	.87	383491	5.96	28
33	962562	.91	362735	5.77	27	33	965772	.87	383849	5.97	27
34	962617	.91	363081	5.77	26	34	965824	.87	384207	5.97	26
35	962672	.91	363428	5.77	25	35	965876	.87	384565	5.97	25
36	962727	.91	363774	5.78	24	36	965929	.87	384923	5.98	24
37	962781	.91	364121	5.78	23	37	965981	.87	385282	5.98	23
38	962836	.91	364468	5.78	22	38	966033	.87	385641	5.98	22
39	962890	.91	364815	5.79	21	39	966085	.86	386000	5.99	21
40	962945	.91	365162	5.79	20	40	966136	.86	386359	5.99	20
41	9.962999	.91	10.365510	5.79	19	41	9.966188	.86	10.386719	6.00	19
42	963054	.91	365857	5.80	18	42	966240	.86	387079	6.00	18
43	963108	.90	366205	5.80	17	43	966292	.86	387439	6.00	17
44	963163	.90	366553	5.80	16	44	966344	.86	387799	6.01	16
45	963217	.90	366901	5.81	15	45	966395	.86	388159	6.01	15
46	963271	.90	367250	5.81	14	46	966447	.86	388520	6.01	14
47	963325	.90	367598	5.81	13	47	966499	.86	388880	6.02	13
48	963379	.90	367947	5.82	12	48	966550	.86	389241	6.02	12
49	963434	.90	368296	5.82	11	49	966602	.86	389603	6.02	11
50	963488	.90	368645	5.82	10	50	966653	.86	389964	6.03	10
51	9.963542	.90	10.368995	5.83	9	51	9.966705	.86	10.390326	6.03	9
52	963596	.90	369344	5.83	8	52	966756	.85	390688	6.03	8
53	963650	.90	369694	5.83	7	53	966808	.85	391050	6.04	7
54	963704	.90	370044	5.83	6	54	966859	.85	391412	6.04	6
55	963757	.90	370394	5.84	5	55	966910	.85	391775	6.04	5
56	963811	.90	370745	5.84	4	56	966961	.85	392137	6.05	4
57	963865	.89	371095	5.85	3	57	967013	.85	392500	6.05	3
58	963919	.89	371446	5.85	2	58	967064	.85	392863	6.06	2
59	963972	.89	371797	5.85	1	59	967115	.85	393227	6.06	1
60	964026		372148		0	60	967166		393590		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.967166	.85	10.393590	6.06	60	0	9.970152	.81	10.415823	6.29	60
1	967217	.85	393954	6.07	59	1	970200	.81	416200	6.30	59
2	967268	.85	394318	6.07	58	2	970249	.81	416578	6.30	58
3	967319	.85	394683	6.07	57	3	970297	.81	416956	6.31	57
4	967370	.85	395047	6.08	56	4	970345	.80	417335	6.31	56
5	967421	.85	395412	6.08	55	5	970394	.80	417714	6.32	55
6	967471	.85	395777	6.09	54	6	970442	.80	418093	6.32	54
7	967522	.84	396142	6.09	53	7	970490	.80	418472	6.32	53
8	967573	.84	396507	6.09	52	8	970538	.80	418851	6.33	52
9	967624	.84	396873	6.10	51	9	970586	.80	419231	6.33	51
10	967674	.84	397239	6.10	50	10	970635	.80	419611	6.34	50
11	9.967725	.84	10.397605	6.10	49	11	9.970683	.80	10.419991	6.34	49
12	967775	.84	397971	6.11	48	12	970731	.80	420371	6.34	48
13	967826	.84	398337	6.11	47	13	970779	.80	420752	6.35	47
14	967876	.84	398704	6.11	46	14	970827	.80	421133	6.35	46
15	967927	.84	399071	6.12	45	15	970874	.80	421514	6.36	45
16	967977	.84	399438	6.12	44	16	970922	.80	421896	6.36	44
17	968027	.84	399806	6.13	43	17	970970	.80	422277	6.36	43
18	968078	.84	400173	6.13	42	18	971018	.80	422659	6.37	42
19	968128	.84	400541	6.13	41	19	971066	.79	423041	6.37	41
20	968178	.84	400909	6.14	40	20	971113	.79	423424	6.38	40
21	9.968228	.84	10.401278	6.14	39	21	9.971161	.79	10.423807	6.38	39
22	968278	.83	401646	6.15	38	22	971208	.79	424190	6.39	38
23	968329	.83	402015	6.15	37	23	971256	.79	424573	6.39	37
24	968379	.83	402384	6.15	36	24	971303	.79	424956	6.39	36
25	968429	.83	402753	6.16	35	25	971351	.79	425340	6.40	35
26	968479	.83	403122	6.16	34	26	971398	.79	425724	6.40	34
27	968528	.83	403492	6.16	33	27	971446	.79	426108	6.41	33
28	968578	.83	403862	6.17	32	28	971493	.79	426493	6.41	32
29	968628	.83	404232	6.17	31	29	971540	.79	426877	6.42	31
30	968678	.83	404602	6.18	30	30	971588	.79	427262	6.42	30
31	9.968728	.83	10.404973	6.18	29	31	9.971635	.79	10.427648	6.42	29
32	968777	.83	405344	6.18	28	32	971682	.79	428033	6.43	28
33	968827	.83	405715	6.18	27	33	971729	.78	428419	6.43	27
34	968877	.83	406086	6.19	26	34	971776	.78	428805	6.44	26
35	968926	.82	406458	6.19	25	35	971823	.78	429191	6.44	25
36	968976	.82	406829	6.20	24	36	971870	.78	429578	6.45	24
37	969025	.82	407201	6.20	23	37	971917	.78	429965	6.45	23
38	969075	.82	407574	6.21	22	38	971964	.78	430352	6.45	22
39	969124	.82	407946	6.21	21	39	972011	.78	430739	6.46	21
40	969173	.82	408319	6.22	20	40	972058	.78	431127	6.46	20
41	9.969223	.82	10.408692	6.22	19	41	9.972105	.78	10.431514	6.47	19
42	969272	.82	409065	6.22	18	42	972151	.78	431902	6.47	18
43	969321	.82	409438	6.23	17	43	972198	.78	432291	6.48	17
44	969370	.82	409812	6.23	16	44	972245	.78	432680	6.48	16
45	969420	.82	410186	6.23	15	45	972291	.78	433068	6.49	15
46	969469	.82	410560	6.24	14	46	972338	.78	433458	6.49	14
47	969518	.82	410934	6.24	13	47	972385	.78	433847	6.49	13
48	969567	.82	411309	6.25	12	48	972431	.77	434237	6.50	12
49	969616	.81	411684	6.25	11	49	972478	.77	434627	6.50	11
50	969665	.81	412059	6.25	10	50	972524	.77	435017	6.51	10
51	9.969714	.81	10.412434	6.26	9	51	9.972570	.77	10.435407	6.51	9
52	969762	.81	412810	6.26	8	52	972617	.77	435798	6.52	8
53	969811	.81	413185	6.27	7	53	972663	.77	436189	6.52	7
54	969860	.81	413561	6.27	6	54	972709	.77	436581	6.53	6
55	969909	.81	413938	6.27	5	55	972755	.77	436972	6.53	5
56	969957	.81	414314	6.28	4	56	972802	.77	437364	6.53	4
57	970006	.81	414691	6.28	3	57	972848	.77	437756	6.54	3
58	970055	.81	415068	6.29	2	58	972894	.77	438149	6.54	2
59	970103	.81	415445	6.29	1	59	972940	.77	438541	6.55	1
60	970152		415823		0	60	972986		438934		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

70°

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.972986	.77	10.438934	6.55	60
1	973032	.76	439327	6.56	59
2	973078	.76	439721	6.56	58
3	973124	.76	440115	6.57	57
4	973169	.76	440509	6.57	56
5	973215	.76	440903	6.58	55
6	973261	.76	441297	6.58	54
7	973307	.76	441692	6.59	53
8	973352	.76	442087	6.59	52
9	973398	.76	442483	6.59	51
10	973444	.76	442879	6.60	50
11	9.973489	.76	10.443275	6.60	49
12	973535	.76	443671	6.61	48
13	973580	.76	444067	6.61	47
14	973625	.76	444464	6.62	46
15	973671	.75	444861	6.62	45
16	973716	.75	445259	6.63	44
17	973761	.75	445656	6.63	43
18	973807	.75	446054	6.64	42
19	973852	.75	446452	6.64	41
20	973897	.75	446851	6.65	40
21	9.973942	.75	10.447250	6.65	39
22	973987	.75	447649	6.65	38
23	974032	.75	448048	6.66	37
24	974077	.75	448448	6.66	36
25	974122	.75	448847	6.67	35
26	974167	.75	449248	6.67	34
27	974212	.75	449648	6.68	33
28	974257	.75	450049	6.68	32
29	974302	.75	450450	6.69	31
30	974347	.74	450851	6.69	30
31	9.974391	.74	10.451253	6.70	29
32	974436	.74	451655	6.70	28
33	974481	.74	452057	6.71	27
34	974525	.74	452460	6.71	26
35	974570	.74	452862	6.72	25
36	974614	.74	453265	6.72	24
37	974659	.74	453669	6.73	23
38	974703	.74	454072	6.73	22
39	974748	.74	454476	6.74	21
40	974792	.74	454881	6.74	20
41	9.974836	.74	10.455285	6.75	19
42	974880	.74	455690	6.75	18
43	974925	.74	456095	6.76	17
44	974969	.73	456501	6.76	16
45	975013	.73	456906	6.77	15
46	975057	.73	457312	6.77	14
47	975101	.73	457719	6.78	13
48	975145	.73	458125	6.78	12
49	975189	.73	458532	6.79	11
50	975233	.73	458939	6.79	10
51	9.975277	.73	10.459347	6.80	9
52	975321	.73	459755	6.80	8
53	975365	.73	460163	6.81	7
54	975408	.73	460571	6.81	6
55	975452	.73	460980	6.82	5
56	975496	.73	461389	6.82	4
57	975539	.73	461798	6.83	3
58	975583	.73	462208	6.83	2
59	975627	.73	462618	6.84	1
60	975670		463028		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

71°

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.975670	.72	10.463028	6.84	60
1	975714	.72	463439	6.85	59
2	975757	.72	463850	6.85	58
3	975800	.72	464261	6.86	57
4	975844	.72	464672	6.86	56
5	975887	.72	465084	6.87	55
6	975930	.72	465496	6.87	54
7	975974	.72	465908	6.88	53
8	976017	.72	466321	6.88	52
9	976060	.72	466734	6.89	51
10	976103	.72	467147	6.89	50
11	9.976146	.72	10.467561	6.90	49
12	976189	.72	467975	6.90	48
13	976232	.71	468389	6.91	47
14	976275	.71	468804	6.91	46
15	976318	.71	469219	6.92	45
16	976361	.71	469634	6.93	44
17	976404	.71	470049	6.93	43
18	976446	.71	470465	6.93	42
19	976489	.71	470881	6.94	41
20	976532	.71	471298	6.95	40
21	9.976574	.71	10.471715	6.95	39
22	976617	.71	472132	6.96	38
23	976660	.71	472549	6.96	37
24	976702	.71	472967	6.97	36
25	976745	.71	473385	6.97	35
26	976787	.71	473803	6.98	34
27	976830	.71	474222	6.98	33
28	976872	.70	474641	6.99	32
29	976914	.70	475060	6.99	31
30	976957	.70	475480	7.00	30
31	9.976999	.70	10.475900	7.01	29
32	977041	.70	476320	7.01	28
33	977083	.70	476741	7.02	27
34	977125	.70	477162	7.02	26
35	977167	.70	477583	7.03	25
36	977209	.70	478005	7.03	24
37	977251	.70	478427	7.03	23
38	977293	.70	478849	7.04	22
39	977335	.70	479272	7.05	21
40	977377	.70	479695	7.05	20
41	9.977419	.70	10.480118	7.06	19
42	977461	.70	480542	7.06	18
43	977503	.70	480966	7.07	17
44	977544	.69	481390	7.08	16
45	977586	.69	481814	7.08	15
46	977628	.69	482239	7.09	14
47	977669	.69	482665	7.09	13
48	977711	.69	483090	7.10	12
49	977752	.69	483516	7.10	11
50	977794	.69	483943	7.11	10
51	9.977835	.69	10.484369	7.12	9
52	977877	.69	484796	7.12	8
53	977918	.69	485223	7.13	7
54	977959	.69	485651	7.13	6
55	978001	.69	486079	7.14	5
56	978042	.69	486507	7.14	4
57	978083	.68	486936	7.15	3
58	978124	.68	487365	7.16	2
59	978165	.68	487794	7.16	1
60	978206		488224		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

72°

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.978206	.68	10.488224	7.17	60
1	978247	.68	488654	7.17	59
2	978288	.68	489084	7.18	58
3	978329	.68	489515	7.18	57
4	978370	.68	489946	7.19	56
5	978411	.68	490378	7.19	55
6	978452	.68	490809	7.20	54
7	978493	.68	491241	7.21	53
8	978533	.68	491674	7.22	52
9	978574	.68	492107	7.22	51
10	978615	.68	492540	7.22	50
11	9.978655	.68	10.492973	7.23	49
12	978696	.67	493407	7.24	48
13	978737	.67	493841	7.24	47
14	978777	.67	494276	7.25	46
15	978817	.67	494711	7.25	45
16	978858	.67	495146	7.26	44
17	978898	.67	495582	7.27	43
18	978939	.67	496018	7.27	42
19	978979	.67	496454	7.28	41
20	979019	.67	496891	7.28	40
21	9.979059	.67	10.497328	7.29	39
22	979100	.67	497765	7.30	38
23	979140	.67	498203	7.30	37
24	979180	.67	498641	7.31	36
25	979220	.67	499080	7.31	35
26	979260	.67	499519	7.32	34
27	979300	.67	499958	7.33	33
28	979340	.66	500397	7.33	32
29	979380	.66	500837	7.34	31
30	979420	.66	501278	7.34	30
31	9.979459	.66	10.501718	7.35	29
32	979499	.66	502159	7.36	28
33	979539	.66	502601	7.36	27
34	979579	.66	503043	7.37	26
35	979618	.66	503485	7.37	25
36	979658	.66	503927	7.38	24
37	979697	.66	504370	7.39	23
38	979737	.66	504814	7.40	22
39	979776	.66	505257	7.40	21
40	979816	.66	505701	7.40	20
41	9.979855	.66	10.506146	7.41	19
42	979895	.66	506590	7.42	18
43	979934	.65	507035	7.43	17
44	979973	.65	507481	7.43	16
45	980012	.65	507927	7.44	15
46	980052	.65	508373	7.44	14
47	980091	.65	508820	7.45	13
48	980130	.65	509267	7.46	12
49	980169	.65	509714	7.46	11
50	980208	.65	510162	7.47	10
51	9.980247	.65	10.510610	7.47	9
52	980286	.65	511059	7.48	8
53	980325	.65	511508	7.49	7
54	980364	.65	511957	7.49	6
55	980403	.65	512407	7.50	5
56	980442	.65	512857	7.51	4
57	980480	.65	513307	7.51	3
58	980519	.64	513758	7.52	2
59	980558	.64	514209	7.53	1
60	980596		514661		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

73°

M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.980596	.64	10.514661	7.53	60
1	980635	.64	515113	7.54	59
2	980673	.64	515565	7.55	58
3	980712	.64	516018	7.55	57
4	980750	.64	516471	7.56	56
5	980789	.64	516925	7.57	55
6	980827	.64	517379	7.57	54
7	980866	.64	517833	7.58	53
8	980904	.64	518288	7.59	52
9	980942	.64	518743	7.60	51
10	980981	.64	519199	7.60	50
11	9.981019	.64	10.519655	7.60	49
12	981057	.64	520111	7.61	48
13	981095	.64	520568	7.62	47
14	981133	.63	521025	7.63	46
15	981171	.63	521483	7.63	45
16	981209	.63	521941	7.64	44
17	981247	.63	522399	7.65	43
18	981285	.63	522858	7.65	42
19	981323	.63	523317	7.66	41
20	981361	.63	523777	7.67	40
21	9.981399	.63	10.524237	7.67	39
22	981436	.63	524697	7.68	38
23	981474	.63	525158	7.69	37
24	981512	.63	525619	7.70	36
25	981549	.63	526081	7.70	35
26	981587	.63	526543	7.70	34
27	981625	.63	527005	7.71	33
28	981662	.63	527468	7.72	32
29	981700	.62	527931	7.73	31
30	981737	.62	528395	7.73	30
31	9.981774	.62	10.528859	7.74	29
32	981812	.62	529324	7.75	28
33	981849	.62	529789	7.75	27
34	981886	.62	530254	7.76	26
35	981924	.62	530720	7.77	25
36	981961	.62	531186	7.78	24
37	981998	.62	531653	7.78	23
38	982035	.62	532120	7.79	22
39	982072	.62	532587	7.80	21
40	982109	.62	533055	7.80	20
41	9.982146	.62	10.533523	7.81	19
42	982183	.62	533992	7.82	18
43	982220	.61	534461	7.83	17
44	982257	.61	534931	7.83	16
45	982294	.61	535401	7.84	15
46	982331	.61	535872	7.85	14
47	982367	.61	536342	7.85	13
48	982404	.61	536814	7.86	12
49	982441	.61	537285	7.87	11
50	982477	.61	537758	7.88	10
51	9.982514	.61	10.538230	7.88	9
52	982551	.61	538703	7.89	8
53	982587	.61	539177	7.90	7
54	982624	.61	539651	7.90	6
55	982660	.61	540125	7.91	5
56	982696	.61	540600	7.92	4
57	982733	.61	541075	7.93	3
58	982769	.60	541551	7.93	2
59	982805	.60	542027	7.94	1
60	982842		542504		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.

74°

M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.982842	.60	10.542504	7.95	60
1	982878	.60	542981	7.96	59
2	982914	.60	543458	7.96	58
3	982950	.60	543936	7.97	57
4	982986	.60	544414	7.98	56
5	983022	.60	544893	7.99	55
6	983058	.60	545372	7.99	54
7	983094	.60	545852	8.00	53
8	983130	.60	546332	8.01	52
9	983166	.60	546813	8.02	51
10	983202	.60	547294	8.02	50
11	9.983238	.60	10.547775	8.03	49
12	983273	.59	548257	8.04	48
13	983309	.59	548740	8.05	47
14	983345	.59	549223	8.06	46
15	983381	.59	549706	8.06	45
16	983416	.59	550190	8.07	44
17	983452	.59	550674	8.08	43
18	983487	.59	551159	8.09	42
19	983523	.59	551644	8.09	41
20	983558	.59	552130	8.10	40
21	9.983594	.59	10.552616	8.11	39
22	983629	.59	553102	8.12	38
23	983664	.59	553589	8.12	37
24	983700	.59	554077	8.13	36
25	983735	.59	554565	8.14	35
26	983770	.59	555053	8.15	34
27	983805	.59	555542	8.16	33
28	983840	.58	556032	8.16	32
29	983875	.58	556521	8.17	31
30	983911	.58	557012	8.18	30
31	9.983946	.58	10.557503	8.19	29
32	983981	.58	557994	8.19	28
33	984015	.58	558486	8.20	27
34	984050	.58	558978	8.21	26
35	984085	.58	559471	8.22	25
36	984120	.58	559964	8.23	24
37	984155	.58	560457	8.23	23
38	984190	.58	560952	8.24	22
39	984224	.58	561446	8.25	21
40	984259	.58	561941	8.26	20
41	9.984294	.58	10.562437	8.27	19
42	984328	.58	562933	8.28	18
43	984363	.58	563430	8.28	17
44	984397	.57	563927	8.29	16
45	984432	.57	564424	8.30	15
46	984466	.57	564922	8.31	14
47	984500	.57	565421	8.32	13
48	984535	.57	565920	8.32	12
49	984569	.57	566420	8.33	11
50	984603	.57	566920	8.34	10
51	9.984638	.57	10.567420	8.35	9
52	984672	.57	567921	8.36	8
53	984706	.57	568423	8.37	7
54	984740	.57	568925	8.38	6
55	984774	.57	569427	8.38	5
56	984808	.57	569930	8.39	4
57	984842	.57	570434	8.40	3
58	984876	.57	570938	8.41	2
59	984910	.57	571442	8.42	1
60	984944		571948		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.

75°

M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.984944	.56	10.571948	8.43	60
1	984978	.56	572453	8.43	59
2	985011	.56	572959	8.44	58
3	985045	.56	573466	8.45	57
4	985079	.56	573973	8.46	56
5	985113	.56	574481	8.47	55
6	985146	.56	574989	8.48	54
7	985180	.56	575497	8.48	53
8	985213	.56	576007	8.49	52
9	985247	.56	576516	8.50	51
10	985280	.56	577026	8.51	50
11	9.985314	.56	10.577537	8.52	49
12	985347	.56	578048	8.53	48
13	985381	.56	578560	8.54	47
14	985414	.55	579073	8.55	46
15	985447	.55	579585	8.55	45
16	985480	.55	580099	8.56	44
17	985514	.55	580613	8.57	43
18	985547	.55	581127	8.58	42
19	985580	.55	581642	8.59	41
20	985613	.55	582158	8.60	40
21	9.985646	.55	10.582674	8.61	39
22	985679	.55	583190	8.62	38
23	985712	.55	583707	8.63	37
24	985745	.55	584225	8.64	36
25	985778	.55	584743	8.64	35
26	985811	.55	585262	8.65	34
27	985843	.55	585781	8.66	33
28	985876	.55	586301	8.67	32
29	985909	.54	586821	8.68	31
30	985942	.54	587342	8.69	30
31	9.985974	.54	10.587863	8.70	29
32	986007	.54	588385	8.71	28
33	986039	.54	588908	8.72	27
34	986072	.54	589431	8.73	26
35	986104	.54	589955	8.74	25
36	986137	.54	590479	8.74	24
37	986169	.54	591004	8.75	23
38	986202	.54	591529	8.76	22
39	986234	.54	592055	8.77	21
40	986266	.54	592581	8.78	20
41	9.986299	.54	10.593108	8.79	19
42	986331	.54	593636	8.80	18
43	986363	.54	594164	8.81	17
44	986395	.53	594692	8.82	16
45	986427	.53	595222	8.83	15
46	986459	.53	595751	8.84	14
47	986491	.53	596282	8.85	13
48	986523	.53	596813	8.86	12
49	986555	.53	597344	8.87	11
50	986587	.53	597876	8.88	10
51	9.986619	.53	10.598409	8.89	9
52	986651	.53	598942	8.90	8
53	986683	.53	599476	8.91	7
54	986714	.53	600010	8.92	6
55	986746	.53	600545	8.93	5
56	986778	.53	601081	8.94	4
57	986809	.53	601617	8.95	3
58	986841	.53	602154	8.96	2
59	986873	.53	602691	8.96	1
60	986904		603229		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.

76°

M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.986904	.52	10.603229	8.97	60
1	986936	.52	603767	8.98	59
2	986967	.52	604306	8.99	58
3	986998	.52	604846	9.00	57
4	987030	.52	605386	9.01	56
5	987061	.52	605927	9.02	55
6	987092	.52	606469	9.03	54
7	987124	.52	607011	9.04	53
8	987155	.52	607553	9.05	52
9	987186	.52	608097	9.06	51
10	987217	.52	608640	9.07	50
11	9.987248	.52	10.609185	9.08	49
12	987279	.52	609730	9.09	48
13	987310	.52	610276	9.10	47
14	987341	.52	610822	9.11	46
15	987372	.51	611369	9.12	45
16	987403	.51	611916	9.13	44
17	987434	.51	612464	9.14	43
18	987465	.51	613013	9.15	42
19	987496	.51	613562	9.17	41
20	987526	.51	614112	9.18	40
21	9.987557	.51	10.614663	9.19	39
22	987588	.51	615214	9.20	38
23	987618	.51	615766	9.21	37
24	987649	.51	616318	9.22	36
25	987679	.51	616871	9.23	35
26	987710	.51	617425	9.24	34
27	987740	.51	617980	9.25	33
28	987771	.51	618534	9.26	32
29	987801	.51	619090	9.27	31
30	987832	.50	619646	9.28	30
31	9.987862	.50	10.620203	9.29	29
32	987892	.50	620761	9.30	28
33	987922	.50	621319	9.31	27
34	987953	.50	621878	9.32	26
35	987983	.50	622437	9.33	25
36	988013	.50	622997	9.34	24
37	988043	.50	623558	9.35	23
38	988073	.50	624119	9.37	22
39	988103	.50	624681	9.38	21
40	988133	.50	625244	9.39	20
41	9.988163	.50	10.625807	9.40	19
42	988193	.50	626371	9.41	18
43	988223	.50	626936	9.42	17
44	988252	.50	627501	9.43	16
45	988282	.50	628067	9.44	15
46	988312	.49	628633	9.45	14
47	988342	.49	629201	9.46	13
48	988371	.49	629768	9.48	12
49	988401	.49	630337	9.49	11
50	988430	.49	630906	9.50	10
51	9.988460	.49	10.631476	9.51	9
52	988489	.49	632047	9.52	8
53	988519	.49	632618	9.53	7
54	988548	.49	633190	9.54	6
55	988578	.49	633763	9.55	5
56	988607	.49	634336	9.57	4
57	988636	.49	634910	9.58	3
58	988666	.49	635485	9.59	2
59	988695	.49	636060	9.60	1
60	988724		636636		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.

77°

M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.988724	.49	10.636636	9.61	60
1	988753	.49	637213	9.62	59
2	988782	.48	637790	9.63	58
3	988811	.48	638368	9.65	57
4	988840	.48	638947	9.66	56
5	988869	.48	639526	9.67	55
6	988898	.48	640107	9.68	54
7	988927	.48	640687	9.69	53
8	988956	.48	641269	9.70	52
9	988985	.48	641851	9.71	51
10	989014	.48	642434	9.73	50
11	9.989042	.48	10.643018	9.74	49
12	989071	.48	643602	9.75	48
13	989100	.48	644187	9.76	47
14	989128	.48	644773	9.77	46
15	989157	.48	645360	9.79	45
16	989186	.48	645947	9.80	44
17	989214	.47	646535	9.81	43
18	989243	.47	647124	9.82	42
19	989271	.47	647713	9.83	41
20	989300	.47	648303	9.85	40
21	9.989328	.47	10.648894	9.86	39
22	989356	.47	649486	9.87	38
23	989385	.47	650078	9.88	37
24	989413	.47	650671	9.90	36
25	989441	.47	651265	9.91	35
26	989469	.47	651859	9.92	34
27	989497	.47	652455	9.93	33
28	989525	.47	653051	9.94	32
29	989553	.47	653647	9.96	31
30	989582	.47	654245	9.97	30
31	9.989610	.47	10.654843	9.98	29
32	989637	.47	655442	9.99	28
33	989665	.46	656042	10.00	27
34	989693	.46	656642	10.02	26
35	989721	.46	657243	10.03	25
36	989749	.46	657845	10.04	24
37	989777	.46	658448	10.06	23
38	989804	.46	659052	10.07	22
39	989832	.46	659656	10.08	21
40	989860	.46	660261	10.10	20
41	9.989887	.46	10.660867	10.11	19
42	989915	.46	661473	10.12	18
43	989942	.46	662081	10.13	17
44	989970	.46	662689	10.15	16
45	989997	.46	663298	10.16	15
46	990025	.46	663907	10.17	14
47	990052	.46	664518	10.19	13
48	990079	.46	665129	10.20	12
49	990107	.45	665741	10.21	11
50	990134	.45	666354	10.23	10
51	9.990161	.45	10.666967	10.24	9
52	990188	.45	667582	10.25	8
53	990215	.45	668197	10.26	7
54	990243	.45	668813	10.28	6
55	990270	.45	669430	10.29	5
56	990297	.45	670047	10.30	4
57	990324	.45	670666	10.32	3
58	990351	.45	671285	10.33	2
59	990378	.45	671905	10.35	1
60	990404		672525		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.990404	.45	10.672525	10.36	60	0	9.991947	.41	10.711348	11.25	60
1	990431	.45	673147	10.37	59	1	991971	.41	712023	11.26	59
2	990458	.45	673769	10.39	58	2	991996	.41	712699	11.28	58
3	990485	.45	674393	10.40	57	3	992020	.41	713376	11.30	57
4	990511	.44	675017	10.41	56	4	992044	.41	714053	11.31	56
5	990538	.44	675642	10.43	55	5	992069	.41	714732	11.33	55
6	990565	.44	676267	10.44	54	6	992093	.41	715412	11.35	54
7	990591	.44	676894	10.45	53	7	992118	.40	716093	11.36	53
8	990618	.44	677521	10.47	52	8	992142	.40	716775	11.38	52
9	990645	.44	678149	10.48	51	9	992166	.40	717458	11.40	51
10	990671	.44	678778	10.50	50	10	992190	.40	718142	11.41	50
11	9.990697	.44	10.679408	10.51	49	11	9.992214	.40	10.718826	11.43	49
12	990724	.44	680039	10.53	48	12	992239	.40	719512	11.45	48
13	990750	.44	680670	10.54	47	13	992263	.40	720199	11.47	47
14	990777	.44	681303	10.55	46	14	992287	.40	720887	11.48	46
15	990803	.44	681936	10.57	45	15	992311	.40	721576	11.50	45
16	990829	.44	682570	10.58	44	16	992335	.40	722266	11.51	44
17	990855	.44	683205	10.60	43	17	992359	.40	722957	11.53	43
18	990882	.44	683841	10.61	42	18	992382	.40	723649	11.55	42
19	990908	.44	684477	10.62	41	19	992406	.40	724342	11.57	41
20	990934	.43	685115	10.64	40	20	992430	.40	725036	11.58	40
21	9.990960	.43	10.685753	10.65	39	21	9.992454	.40	10.725731	11.60	39
22	990986	.43	686392	10.67	38	22	992478	.39	726427	11.62	38
23	991012	.43	687032	10.68	37	23	992501	.39	727124	11.64	37
24	991038	.43	687673	10.70	36	24	992525	.39	727822	11.65	36
25	991064	.43	688315	10.71	35	25	992549	.39	728521	11.67	35
26	991090	.43	688958	10.73	34	26	992572	.39	729221	11.69	34
27	991115	.43	689601	10.74	33	27	992596	.39	729923	11.70	33
28	991141	.43	690246	10.75	32	28	992619	.39	730625	11.72	32
29	991167	.43	690891	10.77	31	29	992643	.39	731329	11.74	31
30	991193	.43	691537	10.78	30	30	992666	.39	732033	11.76	30
31	9.991218	.43	10.692184	10.80	29	31	9.992690	.39	10.732739	11.78	29
32	991244	.43	692832	10.81	28	32	992713	.39	733445	11.79	28
33	991270	.43	693481	10.83	27	33	992736	.39	734153	11.81	27
34	991295	.43	694131	10.84	26	34	992759	.39	734862	11.83	26
35	991321	.43	694782	10.86	25	35	992783	.39	735572	11.85	25
36	991346	.42	695433	10.87	24	36	992806	.39	736283	11.87	24
37	991372	.42	696086	10.89	23	37	992829	.38	736995	11.89	23
38	991397	.42	696739	10.90	22	38	992852	.38	737708	11.90	22
39	991422	.42	697393	10.92	21	39	992875	.38	738422	11.92	21
40	991448	.42	698049	10.93	20	40	992898	.38	739137	11.94	20
41	9.991473	.42	10.698705	10.95	19	41	9.992921	.38	10.739854	11.96	19
42	991498	.42	699362	10.96	18	42	992944	.38	740571	11.98	18
43	991524	.42	700020	10.98	17	43	992967	.38	741290	12.00	17
44	991549	.42	700678	11.00	16	44	992990	.38	742010	12.01	16
45	991574	.42	701338	11.01	15	45	993013	.38	742731	13.03	15
46	991599	.42	701999	11.03	14	46	993036	.38	743453	12.05	14
47	991624	.42	702661	11.04	13	47	993059	.38	744176	12.07	13
48	991649	.42	703323	11.06	12	48	993081	.38	744900	12.09	12
49	991674	.42	703987	11.07	11	49	993104	.38	745626	12.11	11
50	991699	.42	704651	11.09	10	50	993127	.38	746352	12.13	10
51	9.991724	.41	10.705316	11.11	9	51	9.993149	.38	10.747080	12.15	9
52	991749	.41	705983	11.12	8	52	993172	.38	747809	12.17	8
53	991774	.41	706650	11.14	7	53	993195	.38	748539	12.18	7
54	991799	.41	707318	11.15	6	54	993217	.37	749270	12.20	6
55	991823	.41	707987	11.17	5	55	993240	.37	750002	12.22	5
56	991848	.41	708658	11.18	4	56	993262	.37	750736	12.24	4
57	991873	.41	709329	11.20	3	57	993284	.37	751470	12.26	3
58	991897	.41	710001	11.22	2	58	993307	.37	752206	12.28	2
59	991922	.41	710674	11.23	1	59	993329	.37	752943	12.30	1
60	991947		711348		0	60	993351		753681		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.993351	.37	10.753681	12.32	60	0	9.994620	.33	10.800287	13.64	60
1	993374	.37	754421	12.34	59	1	994640	.33	801106	13.66	59
2	993396	.37	755161	12.36	58	2	994660	.33	801926	13.69	58
3	993418	.37	755903	12.38	57	3	994680	.33	802747	13.71	57
4	993440	.37	756646	12.40	56	4	994700	.33	803570	13.74	56
5	993462	.37	757390	12.42	55	5	994720	.33	804394	13.76	55
6	993484	.37	758135	12.44	54	6	994739	.33	805220	13.79	54
7	993506	.37	758882	12.46	53	7	994759	.33	806047	13.81	53
8	993528	.37	759629	12.48	52	8	994779	.33	806876	13.84	52
9	993550	.37	760378	12.50	51	9	994798	.33	807706	13.86	51
10	993572	.37	761128	12.52	50	10	994818	.33	808538	13.89	50
11	9.993594	.36	10.761880	12.54	49	11	9.994838	.33	10.809371	13.91	49
12	993616	.36	762632	12.56	48	12	994857	.33	810206	13.93	48
13	993638	.36	763386	12.58	47	13	994877	.32	811042	13.96	47
14	993660	.36	764141	12.60	46	14	994896	.32	811880	13.99	46
15	993681	.36	764897	12.62	45	15	994916	.32	812720	14.02	45
16	993703	.36	765655	12.65	44	16	994935	.32	813561	14.04	44
17	993725	.36	766414	12.67	43	17	994955	.32	814403	14.07	43
18	993746	.36	767174	12.69	42	18	994974	.32	815248	14.09	42
19	993768	.36	767935	12.71	41	19	994993	.32	816093	14.12	41
20	993789	.36	768698	12.73	40	20	995013	.32	816941	14.15	40
21	9.993811	.36	10.769461	12.75	39	21	9.995032	.32	10.817789	14.17	39
22	993832	.36	770227	12.77	38	22	995051	.32	818640	14.20	38
23	993854	.36	770993	12.79	37	23	995070	.32	819492	14.23	37
24	993875	.36	771761	12.81	36	24	995089	.32	820345	14.25	36
25	993897	.35	772529	12.84	35	25	995108	.32	821201	14.28	35
26	993918	.35	773300	12.86	34	26	995127	.32	822058	14.31	34
27	993939	.35	774071	12.88	33	27	995146	.32	822916	14.33	33
28	993960	.35	774844	12.90	32	28	995165	.32	823776	14.36	32
29	993982	.35	775618	12.92	31	29	995184	.32	824638	14.39	31
30	994003	.35	776393	12.94	30	30	995203	.32	825501	14.42	30
31	9.994024	.35	10.777170	12.97	29	31	9.995222	.31	10.826366	14.44	29
32	994045	.35	777948	12.99	28	32	995241	.31	827233	14.47	28
33	994066	.35	778728	13.01	27	33	995260	.31	828101	14.50	27
34	994087	.35	779508	13.03	26	34	995278	.31	828971	14.53	26
35	994108	.35	780290	13.06	25	35	995297	.31	829843	14.55	25
36	994129	.35	781074	13.08	24	36	995316	.31	830716	14.58	24
37	994150	.35	781858	13.10	23	37	995334	.31	831591	14.61	23
38	994171	.35	782644	13.12	22	38	995353	.31	832468	14.64	22
39	994191	.35	783432	13.15	21	39	995372	.31	833346	14.67	21
40	994212	.35	784220	13.17	20	40	995390	.31	834226	14.70	20
41	9.994233	.35	10.785011	13.19	19	41	9.995409	.31	10.835108	14.73	19
42	994254	.34	785802	13.21	18	42	995427	.31	835992	14.76	18
43	994274	.34	786595	13.24	17	43	995446	.31	836877	14.79	17
44	994295	.34	787389	13.26	16	44	995464	.31	837764	14.81	16
45	994316	.34	788185	13.28	15	45	995482	.31	838653	14.84	15
46	994336	.34	788982	13.31	14	46	995501	.30	839543	14.87	14
47	994357	.34	789780	13.33	13	47	995519	.30	840435	14.90	13
48	994377	.34	790580	13.35	12	48	995537	.30	841329	14.93	12
49	994398	.34	791381	13.38	11	49	995555	.30	842225	14.96	11
50	994418	.34	792183	13.40	10	50	995573	.30	843123	14.99	10
51	9.994438	.34	10.792987	13.42	9	51	9.995591	.30	10.844022	15.02	9
52	994459	.34	793793	13.45	8	52	995610	.30	844923	15.05	8
53	994479	.34	794600	13.47	7	53	995628	.30	845826	15.08	7
54	994499	.34	795408	13.49	6	54	995646	.30	846731	15.11	6
55	994519	.34	796218	13.52	5	55	995664	.30	847637	15.14	5
56	994540	.34	797029	13.54	4	56	995681	.30	848546	15.17	4
57	994560	.33	797841	13.57	3	57	995699	.30	849456	15.20	3
58	994580	.33	798655	13.59	2	58	995717	.30	850368	15.23	2
59	994600	.33	799471	13.61	1	59	995735	.30	851282	15.26	1
60	994620		800287		0	60	995753		852197		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.995753	.29	10.852197	15.29	60	0	9.996751	.26	10.910856	17.43	60
1	995771	.29	853115	15.32	59	1	996766	.26	911902	17.47	59
2	995788	.29	854034	15.35	58	2	996782	.26	912950	17.51	58
3	995806	.29	854956	15.39	57	3	996797	.26	914000	17.55	57
4	995823	.29	855879	15.42	56	4	996812	.26	915053	17.59	56
5	995841	.29	856804	15.45	55	5	996828	.25	916109	17.63	55
6	995859	.29	857731	15.48	54	6	996843	.25	917167	17.67	54
7	995876	.29	858660	15.51	53	7	996858	.25	918227	17.72	53
8	995894	.29	859591	15.55	52	8	996874	.25	919290	17.76	52
9	995911	.29	860524	15.58	51	9	996889	.25	920356	17.80	51
10	995928	.29	861458	15.61	50	10	996904	.25	921424	17.84	50
11	9.995946	.29	10.862395	15.64	49	11	9.996919	.25	10.922495	17.89	49
12	995963	.29	863333	15.67	48	12	996934	.25	923568	17.93	48
13	995980	.29	864274	15.71	47	13	996949	.25	924644	17.97	47
14	995998	.29	865216	15.74	46	14	996964	.25	925722	18.02	46
15	996015	.29	866161	15.77	45	15	996979	.25	926803	18.06	45
16	996032	.29	867107	15.81	44	16	996994	.25	927887	18.10	44
17	996049	.29	868056	15.84	43	17	997009	.25	928973	18.15	43
18	996066	.29	869006	15.87	42	18	997024	.25	930062	18.19	42
19	996083	.28	869959	15.91	41	19	997039	.25	931154	18.24	41
20	996100	.28	870913	15.94	40	20	997053	.25	932248	18.28	40
21	9.996117	.28	10.871870	15.97	39	21	9.997068	.25	10.933345	18.33	39
22	996134	.28	872828	16.01	38	22	997083	.24	934444	18.37	38
23	996151	.28	873789	16.04	37	23	997098	.24	935547	18.42	37
24	996168	.28	874751	16.07	36	24	997112	.24	936652	18.46	36
25	996185	.28	875716	16.11	35	25	997127	.24	937760	18.51	35
26	996202	.28	876683	16.15	34	26	997141	.24	938870	18.55	34
27	996219	.28	877652	16.18	33	27	997156	.24	939984	18.60	33
28	996235	.28	878623	16.22	32	28	997170	.24	941100	18.65	32
29	996252	.28	879596	16.25	31	29	997185	.24	942219	18.70	31
30	996269	.28	880571	16.29	30	30	997199	.24	943341	18.74	30
31	9.996285	.28	10.881548	16.32	29	31	9.997214	.24	10.944465	18.79	29
32	996302	.27	882528	16.36	28	32	997228	.24	945593	18.84	28
33	996318	.27	883509	16.39	27	33	997242	.24	946723	18.89	27
34	996335	.27	884493	16.43	26	34	997257	.24	947856	18.93	26
35	996351	.27	885479	16.46	25	35	997271	.24	948992	18.98	25
36	996368	.27	886467	16.50	24	36	997285	.24	950131	19.03	24
37	996384	.27	887457	16.54	23	37	997299	.24	951273	19.08	23
38	996400	.27	888449	16.58	22	38	997313	.24	952418	19.13	22
39	996417	.27	889444	16.61	21	39	997327	.23	953566	19.18	21
40	996433	.27	890441	16.65	20	40	997341	.23	954716	19.23	20
41	9.996449	.27	10.891440	16.69	19	41	9.997355	.23	10.955870	19.28	19
42	996465	.27	892441	16.72	18	42	997369	.23	957027	19.33	18
43	996482	.27	893444	16.76	17	43	997383	.23	958187	19.38	17
44	996498	.27	894450	16.80	16	44	997397	.23	959349	19.43	16
45	996514	.27	895458	16.84	15	45	997411	.23	960515	19.48	15
46	996530	.27	896468	16.87	14	46	997425	.23	961684	19.53	14
47	996546	.27	897481	16.91	13	47	997439	.23	962856	19.58	13
48	996562	.27	898496	16.95	12	48	997452	.23	964031	19.64	12
49	996578	.26	899513	16.99	11	49	997466	.23	965209	19.69	11
50	996594	.26	900532	17.03	10	50	997480	.23	966391	19.74	10
51	9.996610	.26	10.901554	17.07	9	51	9.997493	.23	10.967575	19.79	9
52	996625	.26	902578	17.11	8	52	997507	.23	968763	19.85	8
53	996641	.26	903605	17.15	7	53	997520	.23	969954	19.90	7
54	996657	.26	904633	17.19	6	54	997534	.22	971148	19.95	6
55	996673	.26	905664	17.22	5	55	997547	.22	972345	20.00	5
56	996688	.26	906698	17.27	4	56	997561	.22	973545	20.06	4
57	996704	.26	907734	17.30	3	57	997574	.22	974749	20.11	3
58	996720	.26	908772	17.34	2	58	997588	.22	975956	20.17	2
59	996735	.26	909813	17.38	1	59	997601	.22	977166	20.23	1
60	996751		910856		0	60	997614		978380		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.997614	.22	10.978380	20.28	60	0	9.998344	.18	11.058048	24.29	60
1	997628	.22	979597	20.33	59	1	998355	.18	059506	24.37	59
2	997641	.22	980817	20.40	58	2	998366	.18	060968	24.45	58
3	997654	.22	982041	20.45	57	3	998377	.18	062435	24.53	57
4	997667	.22	983268	20.51	56	4	998388	.18	063907	24.62	56
5	997680	.22	984498	20.56	55	5	998399	.18	065384	24.70	55
6	997693	.22	985732	20.62	54	6	998410	.18	066866	24.78	54
7	997706	.22	986969	20.68	53	7	998421	.18	068353	24.87	53
8	997719	.22	988210	20.74	52	8	998431	.18	069845	24.95	52
9	997732	.21	989454	20.80	51	9	998442	.18	071342	25.03	51
10	997745	.21	990702	20.85	50	10	998453	.18	072844	25.12	50
11	9.997758	.21	10.991953	20.91	49	11	9.998464	.18	11.074351	25.21	49
12	997771	.21	993208	20.97	48	12	998474	.18	075864	25.30	48
13	997784	.21	994466	21.03	47	13	998485	.18	077381	25.38	47
14	997797	.21	995728	21.09	46	14	998495	.18	078904	25.47	46
15	997809	.21	996993	21.15	45	15	998506	.18	080432	25.56	45
16	997822	.21	998262	21.21	44	16	998516	.17	081966	25.65	44
17	997835	.21	999535	21.27	43	17	998527	.17	083505	25.74	43
18	997847	.21	11.000812	21.34	42	18	998537	.17	085049	25.83	42
19	997860	.21	002092	21.40	41	19	998548	.17	086599	25.92	41
20	997872	.21	003376	21.46	40	20	998558	.17	088154	26.01	40
21	9.997885	.21	11.004663	21.52	39	21	9.998568	.17	11.089715	26.10	39
22	997897	.21	005955	21.58	38	22	998578	.17	091281	26.20	38
23	997910	.21	007250	21.65	37	23	998589	.17	092853	26.29	37
24	997922	.21	008549	21.71	36	24	998599	.17	094430	26.38	36
25	997935	.20	009851	21.78	35	25	998609	.17	096013	26.48	35
26	997947	.20	011158	21.84	34	26	998619	.17	097602	26.58	34
27	997959	.20	012468	21.91	33	27	998629	.17	099197	26.67	33
28	997972	.20	013783	21.97	32	28	998639	.17	100797	26.77	32
29	997984	.20	015101	22.04	31	29	998649	.17	102404	26.87	31
30	997996	.20	016423	22.10	30	30	998659	.17	104016	26.97	30
31	9.998008	.20	11.017749	22.17	29	31	9.998669	.16	11.105634	27.07	29
32	998020	.20	019079	22.23	28	32	998679	.16	107258	27.17	28
33	998032	.20	020414	22.30	27	33	998689	.16	108888	27.27	27
34	998044	.20	021752	22.37	26	34	998699	.16	110524	27.37	26
35	998056	.20	023094	22.44	25	35	998708	.16	112167	27.47	25
36	998068	.20	024440	22.51	24	36	998718	.16	113815	27.58	24
37	998080	.20	025791	22.57	23	37	998728	.16	115470	27.68	23
38	998092	.20	027145	22.65	22	38	998738	.16	117131	27.79	22
39	998104	.20	028504	22.71	21	39	998747	.16	118798	27.89	21
40	998116	.20	029867	22.79	20	40	998757	.16	120471	28.00	20
41	9.998128	.20	11.031234	22.86	19	41	9.998766	.16	11.122151	28.11	19
42	998139	.19	032606	22.93	18	42	998776	.16	123838	28.21	18
43	998151	.19	033981	23.00	17	43	998785	.16	125531	28.32	17
44	998163	.19	035361	23.07	16	44	998795	.16	127230	28.43	16
45	998174	.19	036745	23.14	15	45	998804	.16	128936	28.54	15
46	998186	.19	038134	23.22	14	46	998813	.16	130649	28.66	14
47	998197	.19	039527	23.29	13	47	998823	.15	132368	28.77	13
48	998209	.19	040925	23.37	12	48	998832	.15	134094	28.88	12
49	998220	.19	042326	23.44	11	49	998841	.15	135827	29.00	11
50	998232	.19	043733	23.51	10	50	998851	.15	137567	29.11	10
51	9.998243	.19	11.045144	23.60	9	51	9.998860	.15	11.139314	29.23	9
52	998255	.19	046559	23.66	8	52	998869	.15	141068	29.35	8
53	998266	.19	047979	23.74	7	53	998878	.15	142829	29.46	7
54	998277	.19	049403	23.82	6	54	998887	.15	144597	29.58	6
55	998289	.19	050832	23.90	5	55	998896	.15	146372	29.70	5
56	998300	.19	052266	23.97	4	56	998905	.15	148154	29.82	4
57	998311	.19	053705	24.05	3	57	998914	.15	149943	29.95	3
58	998322	.19	055148	24.13	2	58	998923	.15	151740	30.07	2
59	998333	.18	056596	24.21	1	59	998932	.15	153545	30.19	1
60	998344		058048		0	60	998941		155356		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1″	Tang.	PP1″	M.	M.	Sine.	PP1″	Tang.	PP1″	M.
0	9.998941	.15	11.155356	30.32	60	0	9.999404	.11	11.280604	40.40	60
1	998950	.15	157175	30.45	59	1	999411	.11	283028	40.62	59
2	998958	.15	159002	30.57	58	2	999418	.11	285466	40.85	58
3	998967	.14	160837	30.70	57	3	999424	.11	287917	41.08	57
4	998976	.14	162679	30.83	56	4	999431	.11	290382	41.32	56
5	998984	.14	164529	30.96	55	5	999437	.11	292860	41.55	55
6	998993	.14	166387	31.10	54	6	999443	.11	295354	41.79	54
7	999002	.14	168252	31.23	53	7	999450	.11	297861	42.03	53
8	999010	.14	170126	31.36	52	8	999456	.11	300388	42.28	52
9	999019	.14	172008	31.50	51	9	999463	.10	302919	42.52	51
10	999027	.14	173897	31.63	50	10	999469	.10	305471	42.77	50
11	9.999036	.14	11.175795	31.77	49	11	9.999475	.10	11.308037	43.03	49
12	999044	.14	177702	31.91	48	12	999481	.10	310619	43.28	48
13	999053	.14	179616	32.05	47	13	999487	.10	313216	43.54	47
14	999061	.14	181539	32.19	46	14	999493	.10	315828	43.80	46
15	999069	.14	183471	32.33	45	15	999500	.10	318456	44.07	45
16	999077	.14	185411	32.48	44	16	999506	.10	321100	44.34	44
17	999086	.14	187359	32.62	43	17	999512	.10	323761	44.61	43
18	999094	.13	189317	32.77	42	18	999518	.10	326437	44.88	42
19	999102	.13	191283	32.92	41	19	999524	.10	329130	45.16	41
20	999110	.13	193258	33.07	40	20	999529	.10	331840	45.44	40
21	9.999118	.13	11.195242	33.22	39	21	9.999535	.10	11.334567	45.73	39
22	999126	.13	197235	33.37	38	22	999541	.10	337311	46.02	38
23	999134	.13	199237	33.52	37	23	999547	.10	340072	46.31	37
24	999142	.13	201248	33.68	36	24	999553	.10	342851	46.61	36
25	999150	.13	203269	33.83	35	25	999558	.09	345648	46.91	35
26	999158	.13	205299	33.99	34	26	999564	.09	348463	47.22	34
27	999166	.13	207338	34.15	33	27	999570	.09	351296	47.53	33
28	999174	.13	209387	34.31	32	28	999575	.09	354147	47.84	32
29	999181	.13	211446	34.47	31	29	999581	.09	357018	48.16	31
30	999189	.13	213514	34.64	30	30	999586	.09	359907	48.48	30
31	9.999197	.13	11.215592	34.80	29	31	9.999592	.09	11.362816	48.80	29
32	999205	.13	217680	34.97	28	32	999597	.09	365744	49.13	28
33	999212	.13	219778	35.14	27	33	999603	.09	368692	49.47	27
34	999220	.13	221886	35.31	26	34	999608	.09	371660	49.81	26
35	999227	.13	224005	35.48	25	35	999614	.09	374648	50.15	25
36	999235	.13	226134	35.65	24	36	999619	.09	377657	50.50	24
37	999242	.13	228273	35.83	23	37	999624	.09	380687	50.85	23
38	999250	.12	230422	36.00	22	38	999629	.09	383738	51.21	22
39	999257	.12	232583	36.18	21	39	999635	.09	386811	51.58	21
40	999265	.12	234754	36.36	20	40	999640	.09	389906	51.94	20
41	9.999272	.12	11.236935	36.55	19	41	9.999645	.08	11.393022	52.32	19
42	999279	.12	239128	36.73	18	42	999650	.08	396161	52.70	18
43	999287	.12	241332	36.92	17	43	999655	.08	399323	53.08	17
44	999294	.12	243547	37.10	16	44	999660	.08	402508	53.47	16
45	999301	.12	245773	37.29	15	45	999665	.08	405717	53.87	15
46	999308	.12	248011	37.49	14	46	999670	.08	408949	54.27	14
47	999315	.12	250260	37.68	13	47	999675	.08	412205	54.68	13
48	999322	.12	252521	37.87	12	48	999680	.08	415486	55.10	12
49	999329	.12	254793	38.07	11	49	999685	.08	418792	55.52	11
50	999336	.12	257078	38.27	10	50	999689	.08	422123	55.95	10
51	9.999343	.12	11.259374	38.48	9	51	9.999694	.08	11.425480	56.38	9
52	999350	.12	261683	38.68	8	52	999699	.08	428863	56.82	8
53	999357	.11	264004	38.89	7	53	999704	.08	432273	57.27	7
54	999364	.11	266337	39.09	6	54	999708	.08	435709	57.73	6
55	999371	.11	268683	39.30	5	55	999713	.08	439172	58.19	5
56	999378	.11	271041	39.52	4	56	999717	.08	442664	58.66	4
57	999384	.11	273412	39.74	3	57	999722	.07	446183	59.14	3
58	999391	.11	275796	39.95	2	58	999726	.07	449732	59.62	2
59	999398	.11	278194	40.17	1	59	999731	.07	453309	60.12	1
60	999404		280604		0	60	999735		456916		0
M.	Cosine.	PP1″	Cotang.	PP1″	M.	M.	Cosine.	PP1″	Cotang.	PP1″	M.

M.	Sine.	PP1"	Tang.	PP1"	M.	M.	Sine.	PP1"	Tang.	PP1"	M.
0	9.999735	.07	11.456916	60.62	60	0	9.999934	.04	11.758079	121.7	60
1	999740	.07	460558	61.13	59	1	999936	.04	765379	123.8	59
2	999744	.07	464221	61.65	58	2	999938	.04	772805	125.9	58
3	999748	.07	467920	62.18	57	3	999940	.03	780359	128.1	57
4	999753	.07	471651	62.72	56	4	999942	.03	788047	130.4	56
5	999757	.07	475414	63.26	55	5	999944	.03	795874	132.8	55
6	999761	.07	479210	63.82	54	6	999946	.03	803844	135.3	54
7	999765	.07	483039	64.39	53	7	999948	.03	811964	137.9	53
8	999769	.07	486902	64.96	52	8	999950	.03	820237	140.6	52
9	999774	.07	490800	65.55	51	9	999952	.03	828672	143.3	51
10	999778	.07	494733	66.15	50	10	999954	.03	837273	146.2	50
11	9.999782	.07	11.498702	66.76	49	11	9.999956	.03	11.846048	149.3	49
12	999786	.07	502707	67.38	48	12	999958	.03	855004	152.4	48
13	999790	.07	506750	68.01	47	13	999959	.03	864149	155.7	47
14	999794	.07	510830	68.65	46	14	999961	.03	873490	159.1	46
15	999797	.06	514950	69.31	45	15	999963	.03	883037	162.7	45
16	999801	.06	519108	69.98	44	16	999964	.03	892797	166.4	44
17	999805	.06	523307	70.66	43	17	999966	.03	902783	170.3	43
18	999809	.06	527546	71.35	42	18	999968	.03	913003	174.4	42
19	999813	.06	531828	72.03	41	19	999969	.02	923469	178.7	41
20	999816	.06	536151	72.79	40	20	999971	.02	934194	183.3	40
21	9.999820	.06	11.540519	73.52	39	21	9.999972	.02	11.945191	188.0	39
22	999824	.06	544930	74.28	38	22	999973	.02	956473	193.0	38
23	999827	.06	549387	75.05	37	23	999975	.02	968055	198.3	37
24	999831	.06	553890	75.83	36	24	999976	.02	979956	203.9	36
25	999834	.06	558440	76.63	35	25	999977	.02	992191	209.8	35
26	999838	.06	563038	77.45	34	26	999979	.02	12.004781	216.1	34
27	999841	.06	567685	78.29	33	27	999980	.02	017747	222.7	33
28	999844	.06	572382	79.14	32	28	999981	.02	031111	229.8	32
29	999848	.06	577131	80.02	31	29	999982	.02	044900	237.3	31
30	999851	.05	581932	80.91	30	30	999983	.02	059142	245.4	30
31	9.999854	.05	11.586787	81.82	29	31	9.999985	.02	12.073866	254.0	29
32	999858	.05	591696	82.76	28	32	999986	.02	089106	263.2	28
33	999861	.05	596662	83.71	27	33	999987	.02	104901	273.2	27
34	999864	.05	601685	84.70	26	34	999988	.02	121292	283.9	26
35	999867	.05	606766	85.70	25	35	999989	.02	138326	295.5	25
36	999870	.05	611908	86.72	24	36	999989	.01	156056	308.0	24
37	999873	.05	617111	87.77	23	37	999990	.01	174540	321.7	23
38	999876	.05	622378	88.85	22	38	999991	.01	193845	336.7	22
39	999879	.05	627708	89.95	21	39	999992	.01	214049	353.2	21
40	999882	.05	633105	91.08	20	40	999993	.01	235239	371.2	20
41	9.999885	.05	11.638570	92.24	19	41	9.999993	.01	12.257516	391.3	19
42	999888	.05	644105	93.43	18	42	999994	.01	280997	413.7	18
43	999891	.05	649711	94.65	17	43	999995	.01	305821	438.8	17
44	999894	.05	655390	95.90	16	44	999995	.01	332151	467.1	16
45	999897	.05	661144	97.19	15	45	999996	.01	360180	499.4	15
46	999899	.04	666975	98.51	14	46	999996	.01	390143	536.4	14
47	999902	.04	672886	99.87	13	47	999997	.01	422328	579.4	13
48	999905	.04	678878	101.3	12	48	999997	.01	457091	629.8	12
49	999907	.04	684954	102.7	11	49	999998	.01	494880	689.9	11
50	999910	.04	691116	104.2	10	50	999998	.01	536273	762.6	10
51	9.999913	.04	11.697366	105.7	9	51	9.999999	.01	12.582030	852.5	9
52	999915	.04	703708	107.2	8	52	999999	.00	633183	966.5	8
53	999918	.04	710144	108.9	7	53	999999	.00	691175	1116	7
54	999920	.04	716677	110.5	6	54	999999	.00	758122	1320	6
55	999922	.04	723309	112.2	5	55	10.000000	.00	837304	1615	5
56	999925	.04	730044	114.0	4	56	000000	.00	934214	2082	4
57	999927	.04	736885	115.8	3	57	000000	.00	13.059153	2965	3
58	999929	.04	743835	117.7	2	58	000000	.00	235244	5017	2
59	999932	.04	750898	119.7	1	59	000000		536274		1
60	999934		758079		0	60	000000		Infinite.		0
M.	Cosine.	PP1"	Cotang.	PP1"	M.	M.	Cosine.	PP1"	Cotang.	PP1"	M.

TABLE V.

PRECISE CALCULATION OF FUNCTIONS.

THE proportional parts, as given in Table IV, are sufficient for ordinary use. When precision is desired the following rules should be observed:

I. In finding the logarithmic function of an angle expressed in degrees, minutes, and seconds, derive it from that function which is nearest to it, whether greater or less; for, the proportional parts, being only approximations, should be multiplied by as small a number as possible.

II. In finding the angle from its given function, use that logarithm which differs least from the one given, subtracting or adding as the case may be.

III. To find the logarithmic sine of an angle of less than 2° 36′: reduce it to seconds; add the logarithm of the number of seconds to the logarithmic sine of one second, which is 4.685575; from this sum subtract the difference in the following table corresponding to the number of seconds; the remainder is the required logarithmic sine within one millionth.

IV. Conversely, to find the angle when the given logarithmic sine is less than 8.656702: first, find the angle approximately by Table IV; reduce this to seconds; add to the given sine the difference in the following table corresponding to the number of seconds; from this sum subtract 4.685575; the remainder is the logarithm of the required number of seconds within one.

V. To find the logarithmic tangent of an angle less than 2° 36′: reduce it to seconds; add to the logarithm of the number of seconds the logarithmic tangent of one second, which is 4.685575; to this sum add the difference in the table (p. 419 and 420) corresponding to the number of seconds; the sum is the required logarithmic tangent within one millionth.

VI. To find the angle when the given logarithmic tangent is less than 8.657149, which is the tangent of 2° 36′: first find the angle approximately by Table IV; reduce it to seconds; subtract from the given tangent the difference in the table corresponding to the number of seconds; from this remainder subtract 4.685575; the remainder is the logarithm of the required number of seconds within one.

VII. To find the logarithmic cotangent of an angle less than 2° 36′: reduce it to seconds; subtract the logarithm of the number of seconds from the logarithmic cotangent of one second, which is 15.314425; from this remainder subtract the difference in the table corresponding to the number of seconds; the remainder is the required logarithmic cotangent within one millionth.

VIII. To find the angle when the given logarithmic cotangent is greater than 11.342851, the cotangent of 2° 36′: first find the angle approximately by Table IV; reduce it to seconds; add to the given cotangent the difference in the table corresponding to the number of seconds; subtract this sum from 15.314425; the remainder is the logarithm of the required number of seconds within one.

TABLE V.—AIDS TO

FOR THE SINES OF SMALL ANGLES.

Angles.	Seconds.	Diff.	Angles.	Seconds.	Diff.	Angles.	Seconds.	Diff.
0″	0	0	1° 29′ 50″	5390	50	2° 7′ 30″	7650	100
9′	540	1	30′ 50″	5450	51	8′ 10″	7690	101
15′ 50″	950	2	31′ 40″	5500	52	8′ 45″	7725	102
20′ 20″	1220	3	32′ 30″	5550	53	9′ 20″	7760	103
23′ 50″	1430	4	33′ 30″	5610	54	10′	7800	104
27′	1620	5	34′ 20″	5660	55	10′ 40″	7840	105
29′ 50″	1790	6	35′ 10″	5710	56	11′ 15″	7875	106
32′ 30″	1950	7	36′	5760	57	11′ 50″	7910	107
35′	2100	8	36′ 50″	5810	58	12′ 30″	7950	108
37′ 20″	2240	9	37′ 40″	5860	59	13′ 5″	7985	109
39′ 30″	2370	10	38′ 30″	5910	60	13′ 40″	8020	110
41′ 30″	2490	11	39′ 30″	5970	61	14′ 20″	8060	111
43′ 20″	2600	12	40′ 20″	6020	62	15′	8100	112
45′ 10″	2710	13	41′ 10″	6070	63	15′ 35″	8135	113
47′	2820	14	41′ 50″	6110	64	16′ 10″	8170	114
48′ 40″	2920	15	42′ 40″	6160	65	16′ 45″	8205	115
50′ 20″	3020	16	43′ 30″	6210	66	17′ 20″	8240	116
52′	3120	17	44′ 10″	6250	67	17′ 55″	8275	117
53′ 30″	3210	18	45′	6300	68	18′ 30″	8310	118
55′	3300	19	45′ 50″	6350	69	19′ 5″	8345	119
56′ 30″	3390	20	46′ 30″	6390	70	19′ 40″	8380	120
58′	3480	21	47′ 20″	6440	71	20′ 15″	8415	121
59′ 20″	3560	22	48′	6480	72	20′ 50″	8450	122
1° 00′ 40″	3640	23	48′ 50″	6530	73	21′ 25″	8485	123
2′	3720	24	49′ 30″	6570	74	22′	8520	124
3′ 20″	3800	25	50′ 20″	6620	75	22′ 35″	8555	125
4′ 40″	3880	26	51′	6660	76	23′ 10″	8590	126
5′ 50″	3950	27	51′ 50″	6710	77	23′ 45″	8625	127
7′	4020	28	52′ 30″	6750	78	24′ 20″	8660	128
8′ 10″	4090	29	53′ 10″	6790	79	24′ 55″	8695	129
9′ 20″	4160	30	54′	6840	80	25′ 30″	8730	130
10′ 30″	4230	31	54′ 40″	6880	81	26′	8760	131
11′ 40″	4300	32	55′ 20″	6920	82	26′ 35″	8795	132
12′ 50″	4370	33	56′ 10″	6970	83	27′ 5″	8825	133
14′	4440	34	56′ 50″	7010	84	27′ 40″	8860	134
15′	4500	35	57′ 30″	7050	85	28′ 10″	8890	135
16′ 10″	4570	36	58′ 10″	7090	86	28′ 45″	8925	136
17′ 10″	4630	37	58′ 50″	7130	87	29′ 15″	8955	137
18′ 10″	4690	38	59′ 30″	7170	88	29′ 50″	8990	138
19′ 20″	4760	39	2° 00′ 10″	7210	89	30′ 20″	9020	139
20′ 20″	4820	40	50″	7250	90	30′ 55″	9055	140
21′ 20″	4880	41	1′ 40″	7300	91	31′ 25″	9085	141
22′ 20″	4940	42	2′ 20″	7340	92	32′	9120	142
23′ 20″	5000	43	3′	7380	93	32′ 30″	9150	143
24′ 20″	5060	44	3′ 35″	7415	94	33′ 5″	9185	144
25′ 10″	5110	45	4′ 10″	7450	95	33′ 35″	9215	145
26′ 10″	5170	46	4′ 50″	7490	96	34′ 5″	9245	146
27′ 10″	5230	47	5′ 30″	7530	97	34′ 40″	9280	147
28′ 10″	5290	48	6′ 10″	7570	98	35′ 10″	9310	148
29′	5340	49	6′ 50″	7610	99	35′ 40″	9340	149
29′ 50″	5390		7′ 30″	7650		36′ 15″	9375	

PRECISE CALCULATIONS.

FOR TANGENTS AND COTANGENTS OF SMALL ANGLES.

Angles.	Seconds.	Diff.	Angles.	Seconds.	Diff.	Angles.	Seconds.	Diff.
0″	0	0	1° 3′ 30″	3810	50	1° 30′ 10″	5410	100
7′ 10″	430	1	4′ 10″	3850	51	30′ 30″	5430	101
11′ 10″	670	2	4′ 50″	3890	52	31′	5460	102
14′ 10″	850	3	5′ 30″	3930	53	31′ 30″	5490	103
17′	1020	4	6′	3960	54	32′	5520	104
19′	1140	5	6′ 40″	4000	55	32′ 20″	5540	105
21′	1260	6	7′ 20″	4040	56	32′ 50″	5570	106
23′	1380	7	7′ 50″	4070	57	33′ 10″	5590	107
24′ 50″	1490	8	8′ 30″	4110	58	33′ 40″	5620	108
26′ 30″	1590	9	9′	4140	59	34′ 10″	5650	109
27′ 50″	1670	10	9′ 40″	4180	60	34′ 30″	5670	110
29′ 20″	1760	11	10′ 20″	4220	61	35′	5700	111
30′ 40″	1840	12	10′ 50″	4250	62	35′ 20″	5720	112
32′	1920	13	11′ 30″	4290	63	35′ 50″	5750	113
33′ 10″	1990	14	12′	4320	64	36′ 10″	5770	114
34′ 20″	2060	15	12′ 30″	4350	65	36′ 40″	5800	115
35′ 30″	2130	16	13′ 10″	4390	66	37′ 10″	5830	116
36′ 40″	2200	17	13′ 40″	4420	67	37′ 30″	5850	117
37′ 50″	2270	18	14′ 10″	4450	68	38′	5880	118
38′ 50″	2330	19	14′ 50″	4490	69	38′ 20″	5900	119
39′ 50″	2390	20	15′ 20″	4520	70	38′ 50″	5930	120
40′ 50″	2450	21	15′ 50″	4550	71	39′ 10″	5950	121
41′ 50″	2510	22	16′ 20″	4580	72	39′ 30″	5970	122
42′ 50″	2570	23	17′	4620	73	40′	6000	123
43′ 50″	2630	24	17′ 30″	4650	74	40′ 20″	6020	124
44′ 40″	2680	25	18′	4680	75	40′ 50″	6050	125
45′ 40″	2740	26	18′ 30″	4710	76	41′ 10″	6070	126
46′ 30″	2790	27	19′	4740	77	41′ 40″	6100	127
47′ 20″	2840	28	19′ 30″	4770	78	42′	6120	128
48′ 10″	2890	29	20′	4800	79	42′ 30″	6150	129
49′	2940	30	20′ 30″	4830	80	42′ 50″	6170	130
49′ 50″	2990	31	21′	4860	81	43′ 10″	6190	131
50′ 40″	3040	32	21′ 30″	4890	82	43′ 40″	6220	132
51′ 30″	3090	33	22′	4920	83	44′	6240	133
52′ 20″	3140	34	22′ 30″	4950	84	44′ 30″	6270	134
53′	3180	35	23′	4980	85	44′ 50″	6290	135
53′ 50″	3230	36	23′ 30″	5010	86	45′ 20″	6320	136
54′ 40″	3280	37	24′	5040	87	45′ 40″	6340	137
55′ 20″	3320	38	24′ 30″	5070	88	46′	6360	138
56′	3360	39	25′	5100	89	46′ 20″	6380	139
56′ 50″	3410	40	25′ 30″	5130	90	46′ 40″	6400	140
57′ 30″	3450	41	26′	5160	91	47′ 10″	6430	141
58′ 10″	3490	42	26′ 30″	5190	92	47′ 30″	6450	142
58′ 50″	3530	43	26′ 50″	5210	93	48′	6480	143
59′ 30″	3570	44	27′ 20″	5240	94	48′ 20″	6500	144
0′ 20″	3620	45	27′ 50″	5270	95	48′ 40″	6520	145
1′	3660	46	28′ 20″	5300	96	49′	6540	146
1′ 40″	3700	47	28′ 40″	5320	97	49′ 20″	6560	147
2′ 10″	3730	48	29′ 10″	5350	98	49′ 40″	6580	148
2′ 50″	3770	49	29′ 40″	5380	99	50′ 10″	6610	149
3′ 30″	3810		30′ 10″	5410		50′ 30″	6630	

TABLE V.—AIDS TO PRECISE CALCULATIONS.

FOR TANGENTS AND COTANGENTS OF SMALL ANGLES.

Angles.	Seconds.	Diff.	Angles.	Seconds.	Diff.	Angles	Seconds.	Diff.
1° 50′ 30″	6630	150	2° 7′ 40″	7660	200	2° 22′ 35″	8555	250
50′ 50″	6650	151	8′	7680	201	22′ 55″	8575	251
51′ 10″	6670	152	8′ 15″	7695	202	23′ 10″	8590	252
51′ 30″	6690	153	8′ 30″	7710	203	23′ 30″	8610	253
52′	6720	154	8′ 50″	7730	204	23′ 45″	8625	254
52′ 20″	6740	155	9′ 10″	7750	205	24′	8640	255
52′ 40″	6760	156	9′ 30″	7770	206	24′ 20″	8660	256
53′	6780	157	9′ 50″	7790	207	24′ 35″	8675	257
53′ 20″	6800	158	10′ 10″	7810	208	24′ 55″	8695	258
53′ 50″	6830	159	10′ 20″	7820	209	25′ 10″	8710	259
54′ 10″	6850	160	10′ 40″	7840	210	25′ 25″	8725	260
54′ 30″	6870	161	10′ 55″	7855	211	25′ 45″	8745	261
54′ 50″	6890	162	11′ 15″	7875	212	26′	8760	262
55′ 10″	6910	163	11′ 35″	7895	213	26′ 20″	8780	263
55′ 30″	6930	164	11′ 55″	7915	214	26′ 35″	8795	264
55′ 50″	6950	165	12′ 15″	7935	215	26′ 50″	8810	265
56′ 10″	6970	166	12′ 35″	7955	216	27′ 10″	8830	266
56′ 30″	6990	167	12′ 55″	7975	217	27′ 25″	8845	267
56′ 50″	7010	168	13′ 15″	7995	218	27′ 45″	8865	268
57′ 10″	7030	169	13′ 35″	8015	219	28′	8880	269
57′ 40″	7060	170	13′ 50″	8030	220	28′ 15″	8895	270
58′	7080	171	14′ 10″	8050	221	28′ 35″	8915	271
58′ 20″	7100	172	14′ 30″	8070	222	28′ 50″	8930	272
58′ 40″	7120	173	14′ 45″	8085	223	29′ 10″	8950	273
59′	7140	174	15′ 5″	8105	224	29′ 25″	8965	274
59′ 20″	7160	175	15′ 20″	8120	225	29′ 40″	8980	275
59′ 40″	7180	176	15′ 40″	8140	226	30′	9000	276
2° 00′ 00″	7200	177	15′ 55″	8155	227	30′ 15″	9015	277
20″	7220	178	16′ 15″	8175	228	30′ 30″	9030	278
40″	7240	179	16′ 30″	8190	229	30′ 50″	9050	279
1′	7260	180	16′ 50″	8210	230	31′ 5″	9065	280
1′ 20″	7280	181	17′ 5″	8225	231	31′ 20″	9080	281
1′ 40″	7300	182	17′ 25″	8245	232	31′ 35″	9095	282
2′	7320	183	17′ 40″	8260	233	31′ 55″	9115	283
2′ 20″	7340	184	18′	8280	234	32′ 10″	9130	284
2′ 40″	7360	185	18′ 15″	8295	235	32′ 25″	9145	285
3′	7380	186	18′ 35″	8315	236	32′ 40″	9160	286
3′ 20″	7400	187	18′ 55″	8335	237	32′ 55″	9175	287
3′ 40″	7420	188	19′ 15″	8355	238	33′ 15″	9195	288
4′	7440	189	19′ 30″	8370	239	33′ 30″	9210	289
4′ 20″	7460	190	19′ 45″	8385	240	33′ 45″	9225	290
4′ 40″	7480	191	20′ 5″	8405	241	34′	9240	291
5′	7500	192	20′ 20″	8420	242	34′ 15″	9255	292
5′ 20″	7520	193	20′ 40″	8440	243	34′ 30″	9270	293
5′ 40″	7540	194	20′ 55″	8455	244	34′ 45″	9285	294
6′	7560	195	21′ 15″	8475	245	35′	9300	295
6′ 20″	7580	196	21′ 30″	8490	246	35′ 20″	9320	296
6′ 40″	7600	197	21′ 45″	8505	247	35′ 35″	9335	297
7′	7620	198	22′ 5″	8525	248	35′ 50″	9350	298
7′ 20″	7640	199	22′ 20″	8540	249	36′ 5″	9365	299
7′ 40″	7660		22′ 35″	8555		36′ 20″	9380	